Polyhedral Complex Derivation from Piecewise Trilinear Networks

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github.com/naver-ai/tropical-nerf.pytorch
naver-ai.github.io/tropical-nerf

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Signed distance function (SDF)

- **Definition.** If Ω is a subset of a space X with a metric d, the SDF f is defined as:

$$f(x) = \begin{cases} -d(x, \partial\Omega) & \text{if } x \in \Omega \\ d(x, \partial\Omega) & \text{if } x \in \Omega^{\complement} \end{cases}$$

where $\partial\Omega$ denotes the boundary of Ω, and the metric with the boundary is:

$$d(x, \partial\Omega) := \inf_{y \in \partial\Omega} d(x, y),\ x^{\forall} \in X$$

- In the Euclidean space, d is the shortest distance from x to the boundary.

https://en.wikipedia.org/wiki/Signed_distance_function; A slightly different definition is used for consistency.

Properties in Euclidean space

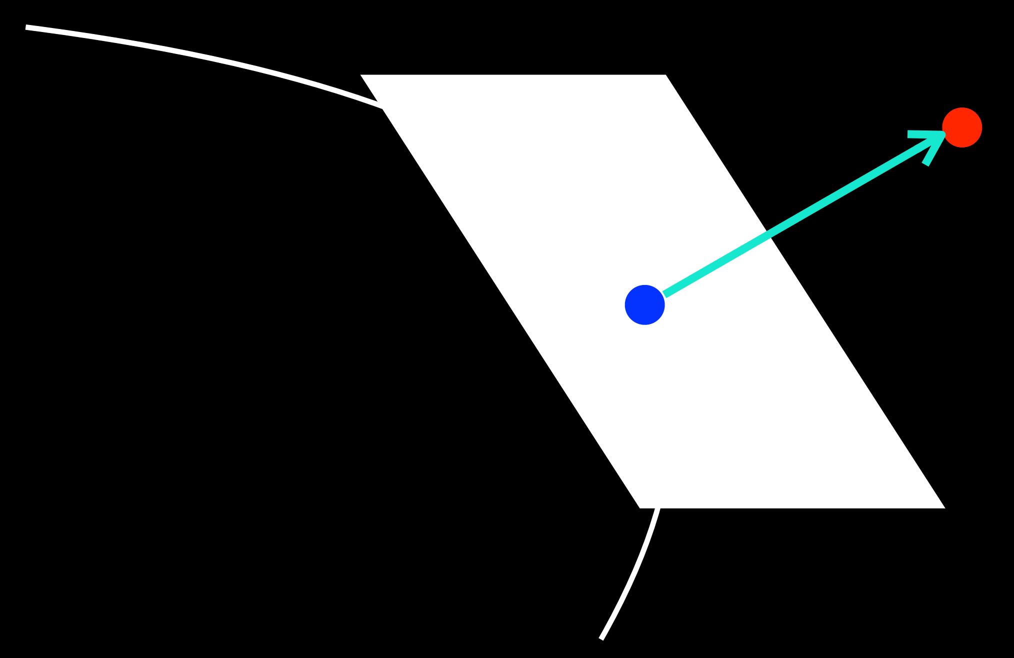
- For the Euclidean space with piecewise smooth boundary, the SDF is differentiable *almost everywhere*, and its gradient satisfies the *eikonal* equation:

$$|\nabla f| = 1$$

- Particularly, the gradient of f on the boundary of Ω is the *outward* normal vector:

$$\nabla f(x) = N(x).$$

- Therefore, the SDF is *a differentiable extension* of the normal vector field.



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Mesh from a SDF

- If an SDF is made up of ReLU-based neural networks, we can extract a mesh from the networks by leveraging *continuous piecewise affine (CPWA)* properties.
- We utilize the fact that 1) ReLU activation patterns create distinct linear regions, and 2) each neuron represents a folded hyperplane across these regions.

  👉 Formal descriptions using *tropical geometry* can be found in Sec. 3 & Appendix A.
- Since the number of linear regions *exponentially* grows with the depth of networks, Edge subdivision (Berzins, 2023) is an optimal algorithm that iterates over neurons, *not over the linear regions*, while subdividing the current set of edges.

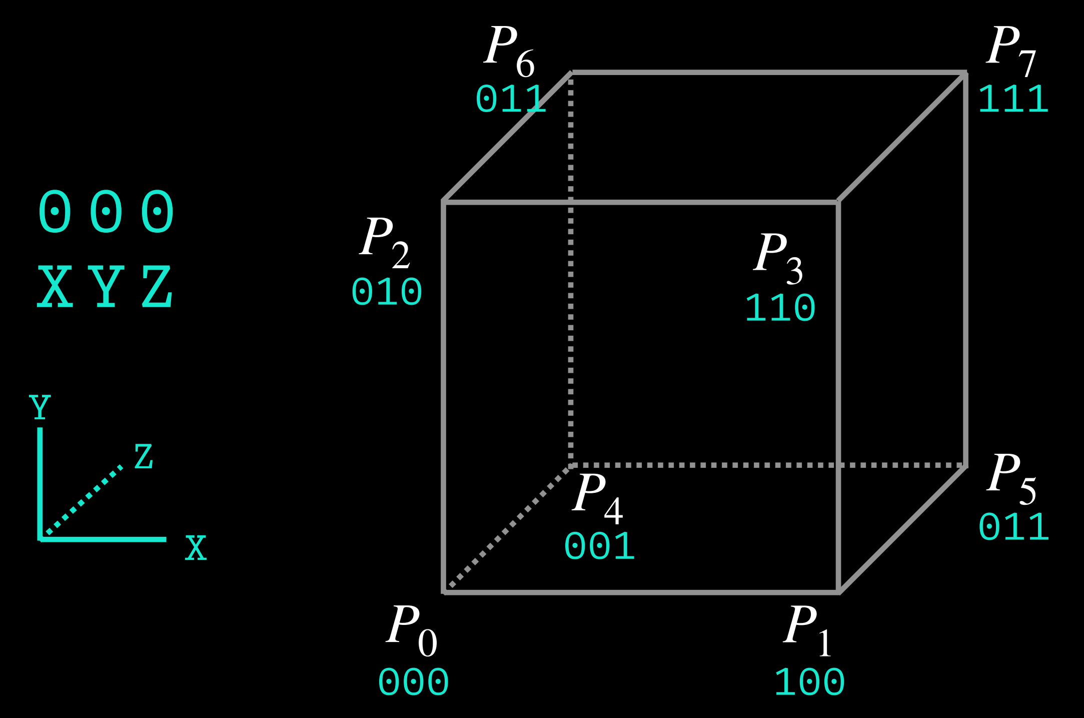
Motivation

- HashGrid (Müller et al., 2022) exploit trilinear interpolation to achieve fast convergence and mitigate spectral bias.
- Can we still analytically extract 3D mesh from the learned SDF?
  - 👉 *Eikonal constraint* makes the parameterized trilinear interpolation *continuous piecewise affine (CPWA)* function (Thm. 4.5 & Coro. 4.6).
  - 👉 Small high-resolution grids further reduce approximation errors since they can better fit curves with finer linear segments.
- For discussion, we define $\tau(\mathrm{x})$ as the HashGrid function with a single output.

Hypersurface and eikonal constraint

Theorem 4.5 (Hypersurface and eikonal constraint)**.** A hypersurface $\tau(\mathrm{x}) = 0$ passing two points $\tau(\mathrm{x}_0) = \tau(\mathrm{x}_7) = 0$ while $\tau(\mathrm{x}_{1\ldots6}) \neq 0$ for the remaining six points. These points form a cube, with x_0 and x_7 positioned on the diagonal of the cube. The hypersurface satisfies the eikonal constraint $\|\nabla\tau(\mathrm{x})\|_2^2 = 1$ for all $\mathrm{x} \in [0,\ 1]^3$. Then, the hypersurface of $\tau(\mathrm{x}) = 0$ is a plane.

Proof sketch. The eikonal constraint makes the surfaces of an SDF smooth and coherent structure. The linearity would make planar surfaces. For the proof, we calculate the *second derivatives of trilinear interpolation to be zeros* and find the constraints to satisfy the linearity.



Implications of Thm. 4.5

- To satisfy the eikonal constraint, the following equations are true:

$$\left.\begin{aligned} \tau(\mathrm{x}_1) + \tau(\mathrm{x}_6) &= 0 \\ \tau(\mathrm{x}_2) + \tau(\mathrm{x}_5) &= 0 \\ \tau(\mathrm{x}_3) + \tau(\mathrm{x}_4) &= 0 \end{aligned}\right\} \textit{opposite two vertices}$$

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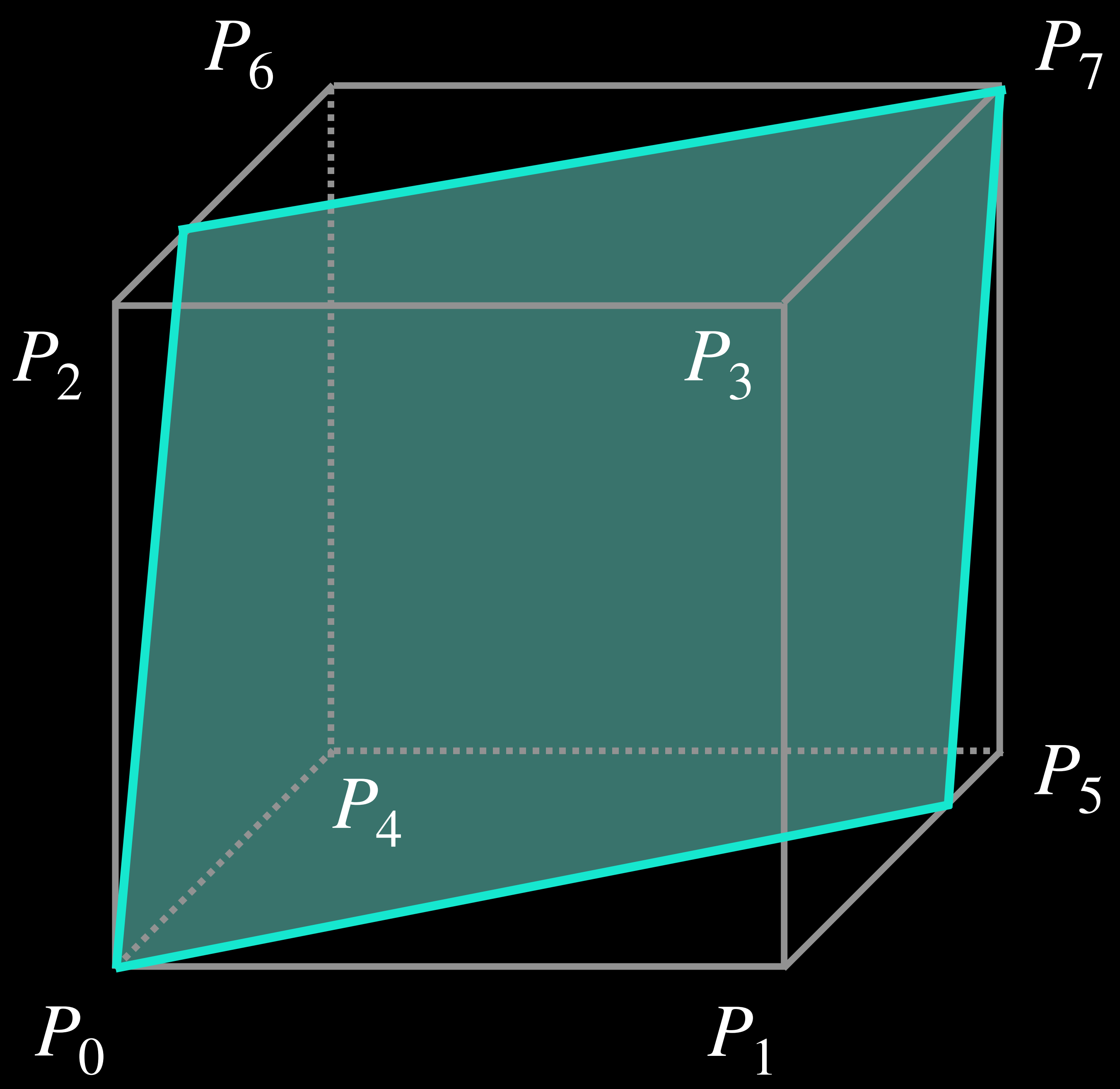
where the hash table entries are $P_i = \tau(\mathrm{x}_i)$.



Ref. Appendix D Theoretical proofs

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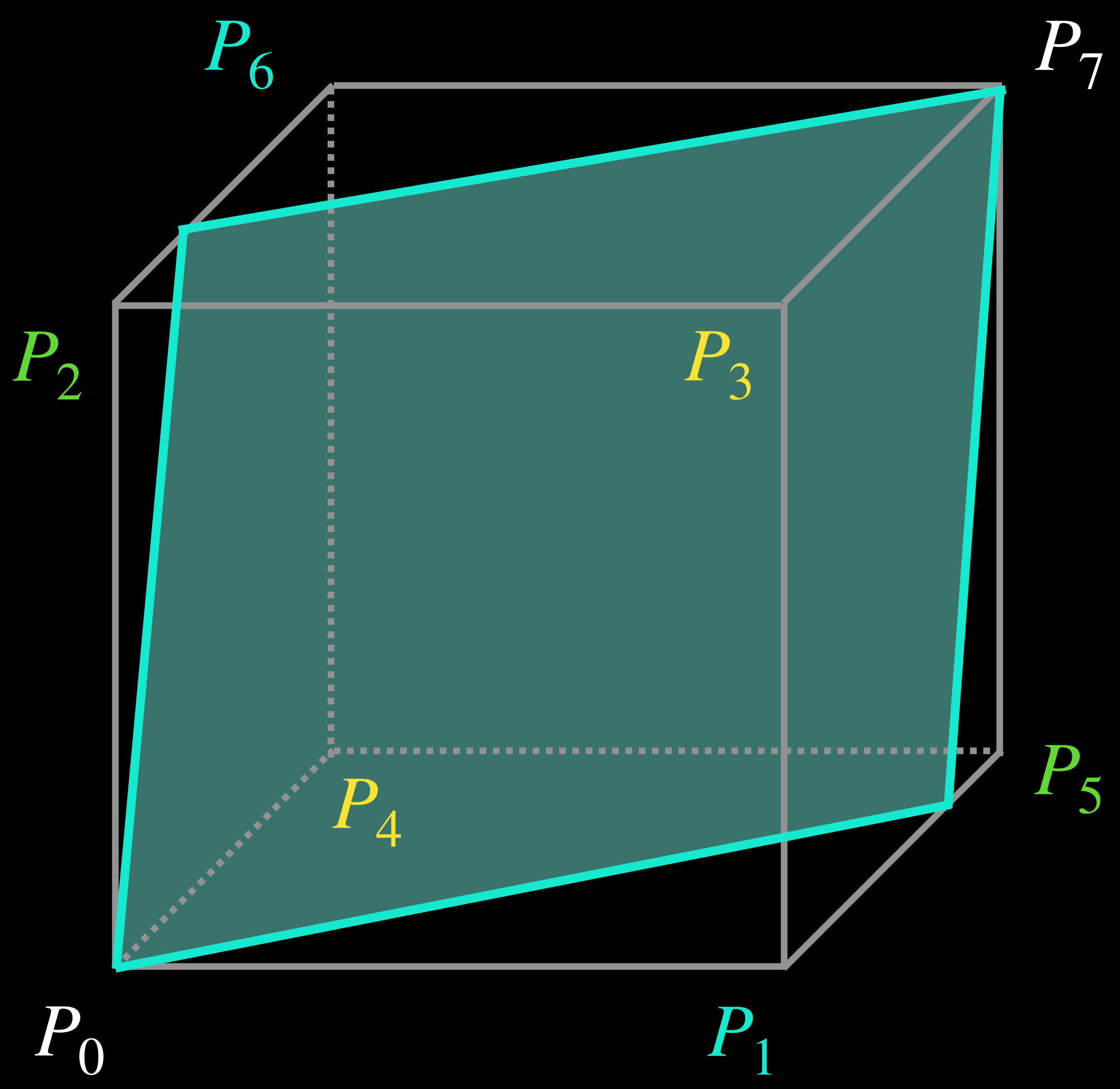

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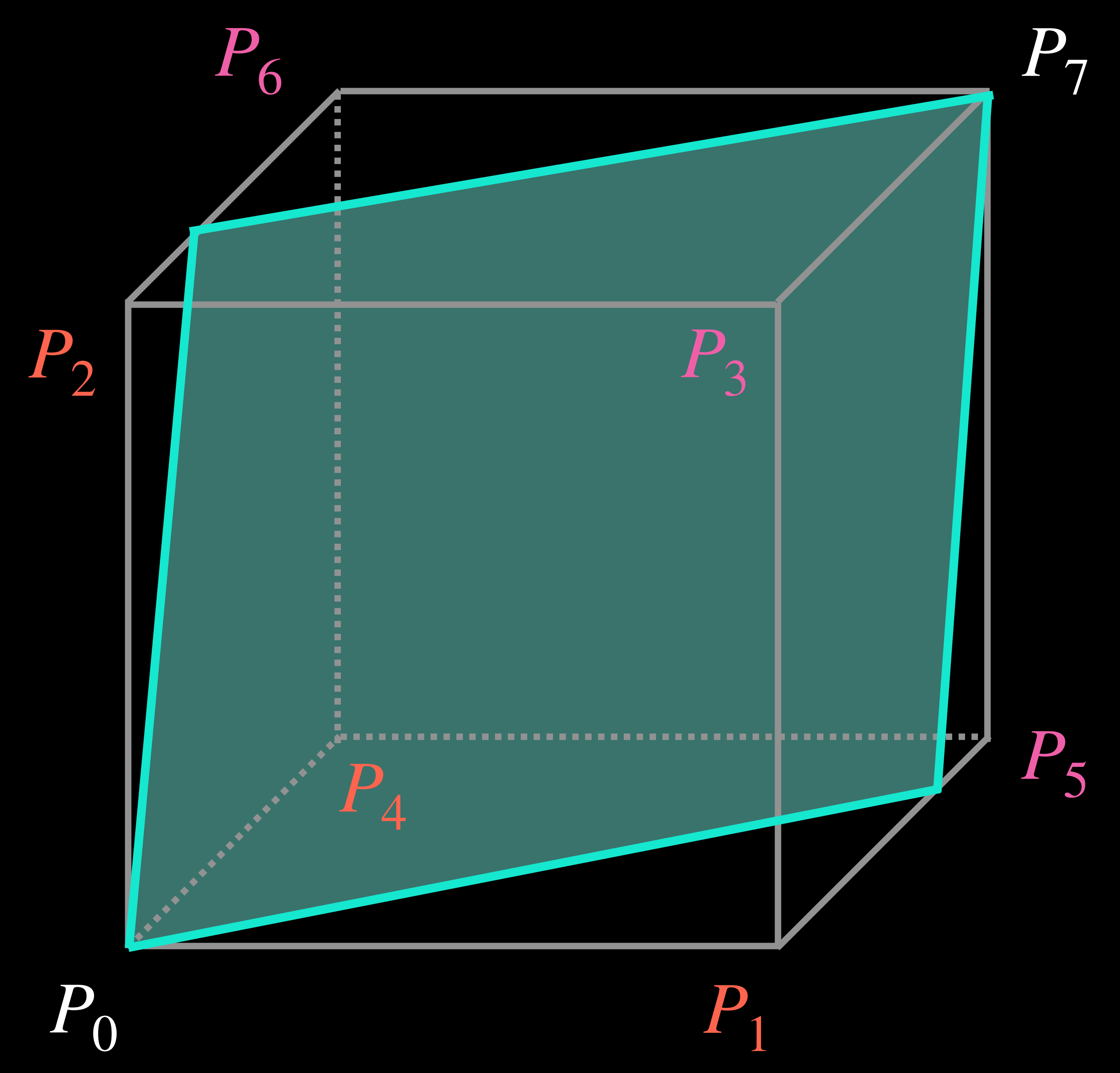

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Flatness error

- To satisfy the eikonal constraint, the following equations are true:

$$\left.\begin{aligned} &\text{●}\ \ \tau(\mathrm{x}_1) + \tau(\mathrm{x}_6) = 0 \\ &\text{▲}\ \ \tau(\mathrm{x}_2) + \tau(\mathrm{x}_5) = 0 \\ &\text{■}\ \ \tau(\mathrm{x}_3) + \tau(\mathrm{x}_4) = 0 \end{aligned}\right\}\ \textit{opposite two vertices}$$

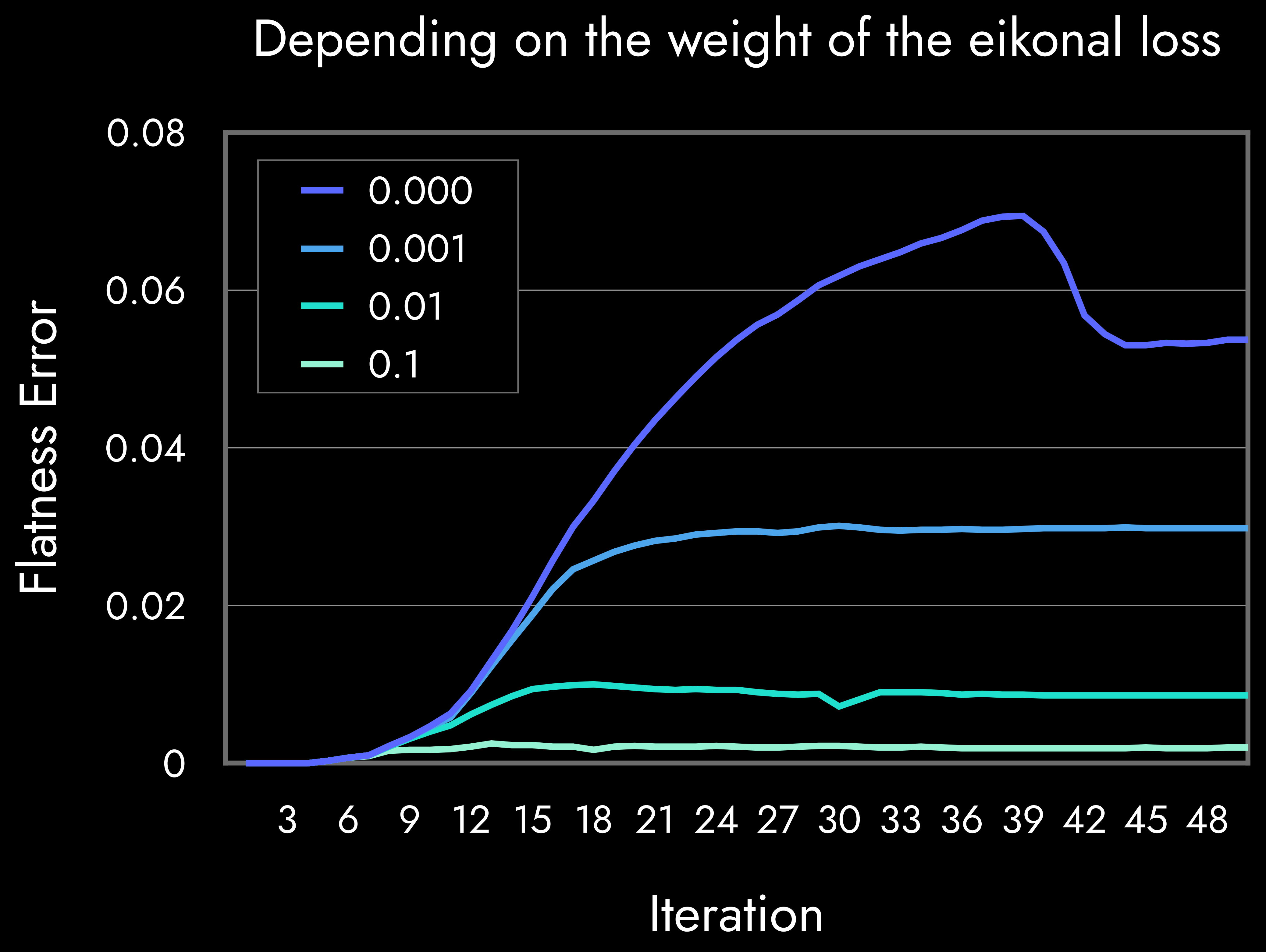
$$\left.\begin{aligned} &\text{◆}\ \ \tau(\mathrm{x}_1) + \tau(\mathrm{x}_2) + \tau(\mathrm{x}_4) = 0 \\ &\text{★}\ \ \tau(\mathrm{x}_3) + \tau(\mathrm{x}_5) + \tau(\mathrm{x}_6) = 0 \end{aligned}\right\}\ \textit{two diagonal groups}$$

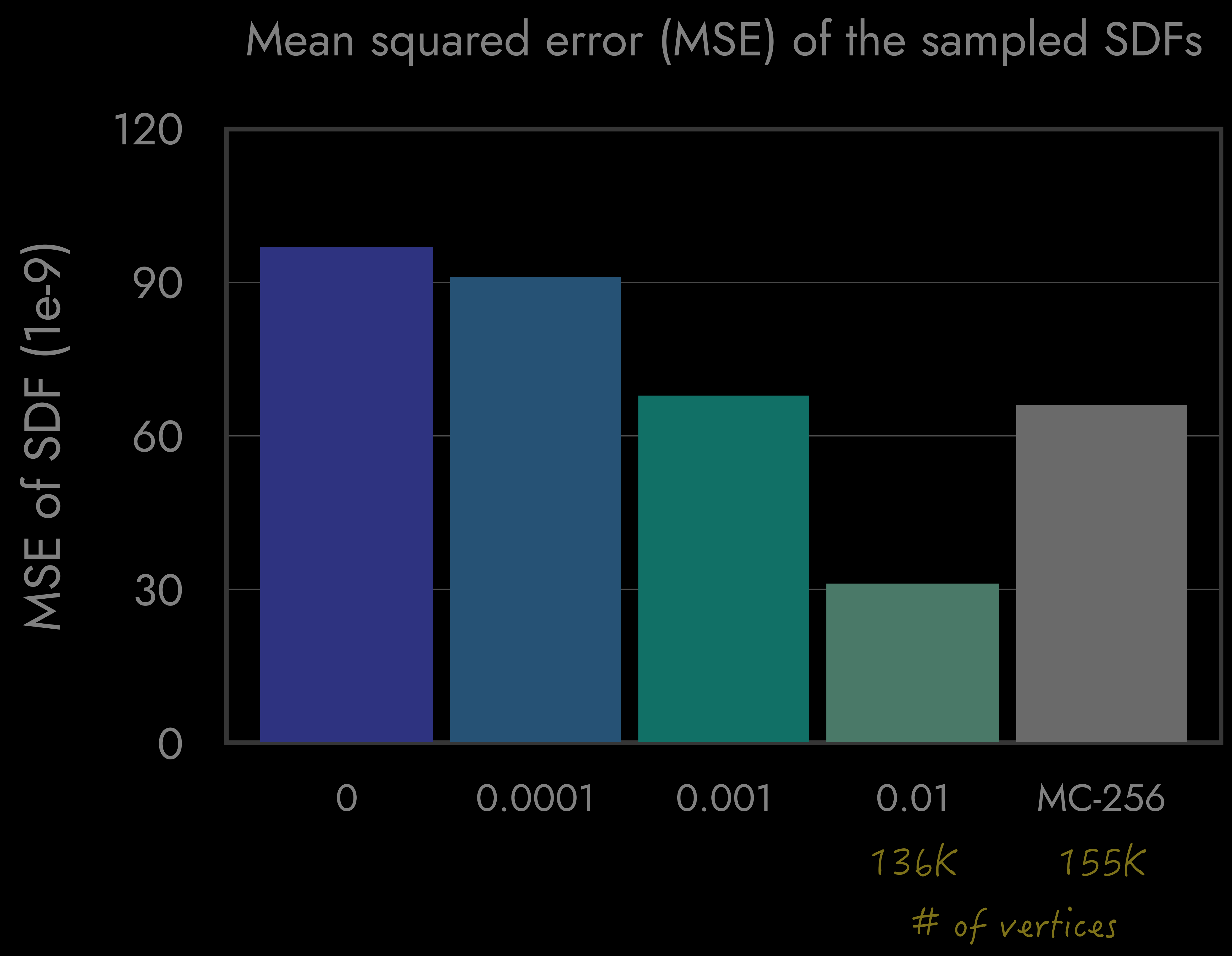
- The mean absolute error (MAE) of *flatness* is defined as:

$$\mathbb{E}_{\mathscr{E}}\left[\frac{1}{6}\left(\lVert \text{●} \rVert_1 + \lVert \text{▲} \rVert_1 + \lVert \text{■} \rVert_1\right) + \frac{1}{4}\left(\lVert \text{◆} \rVert_1 + \lVert \text{★} \rVert_1\right)\right].$$

Ref. Appendix D Theoretical proofs

Empirical validations





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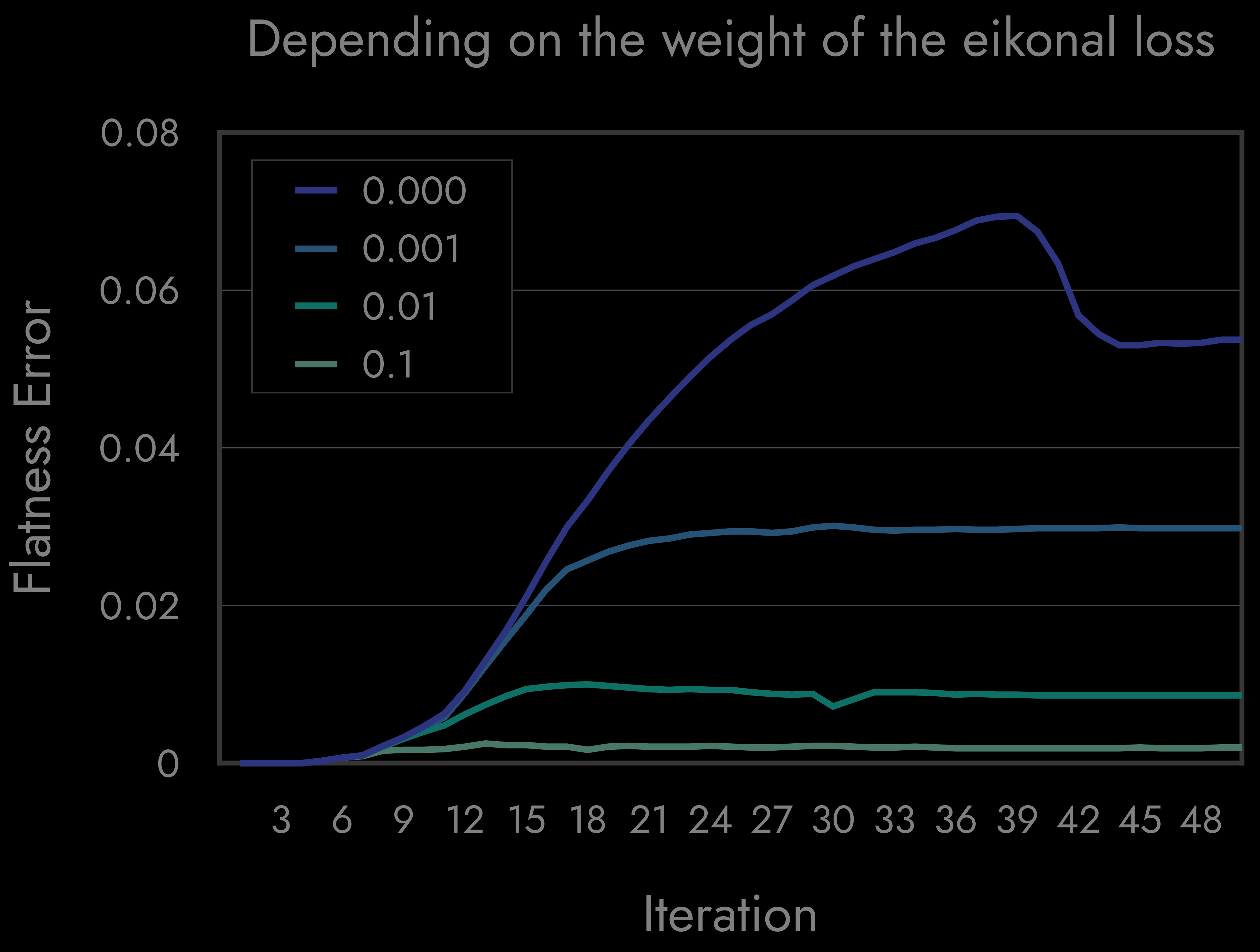
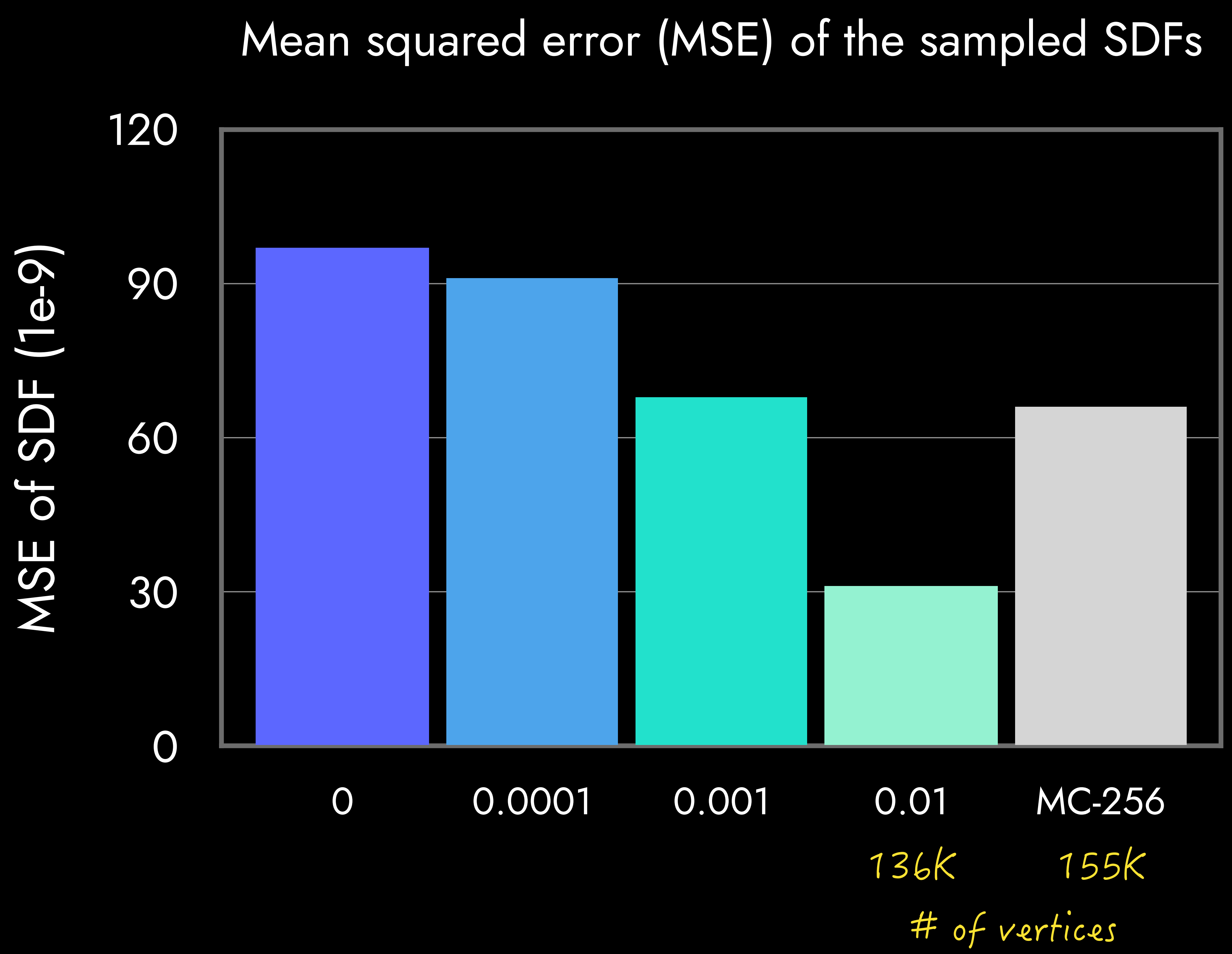
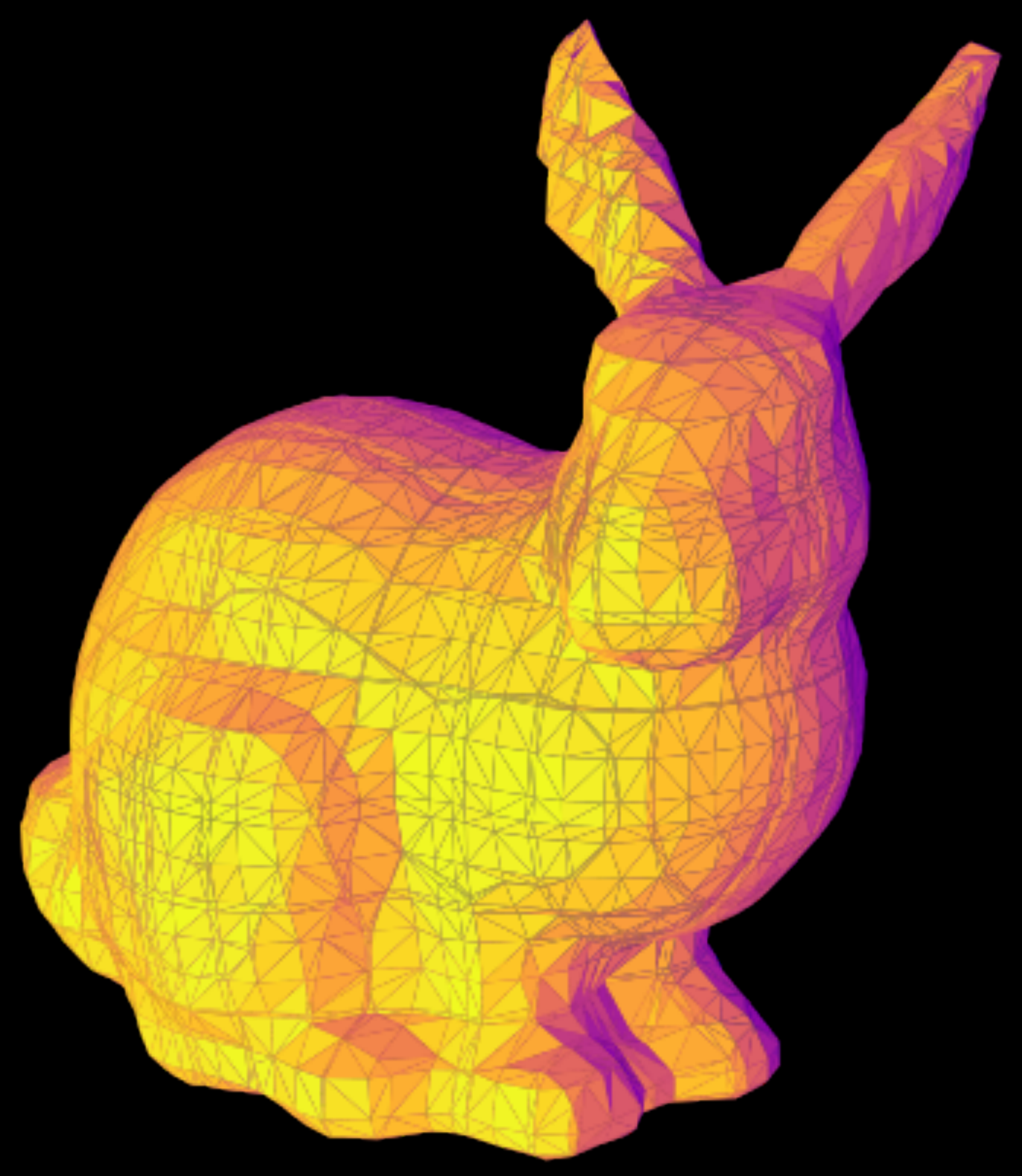




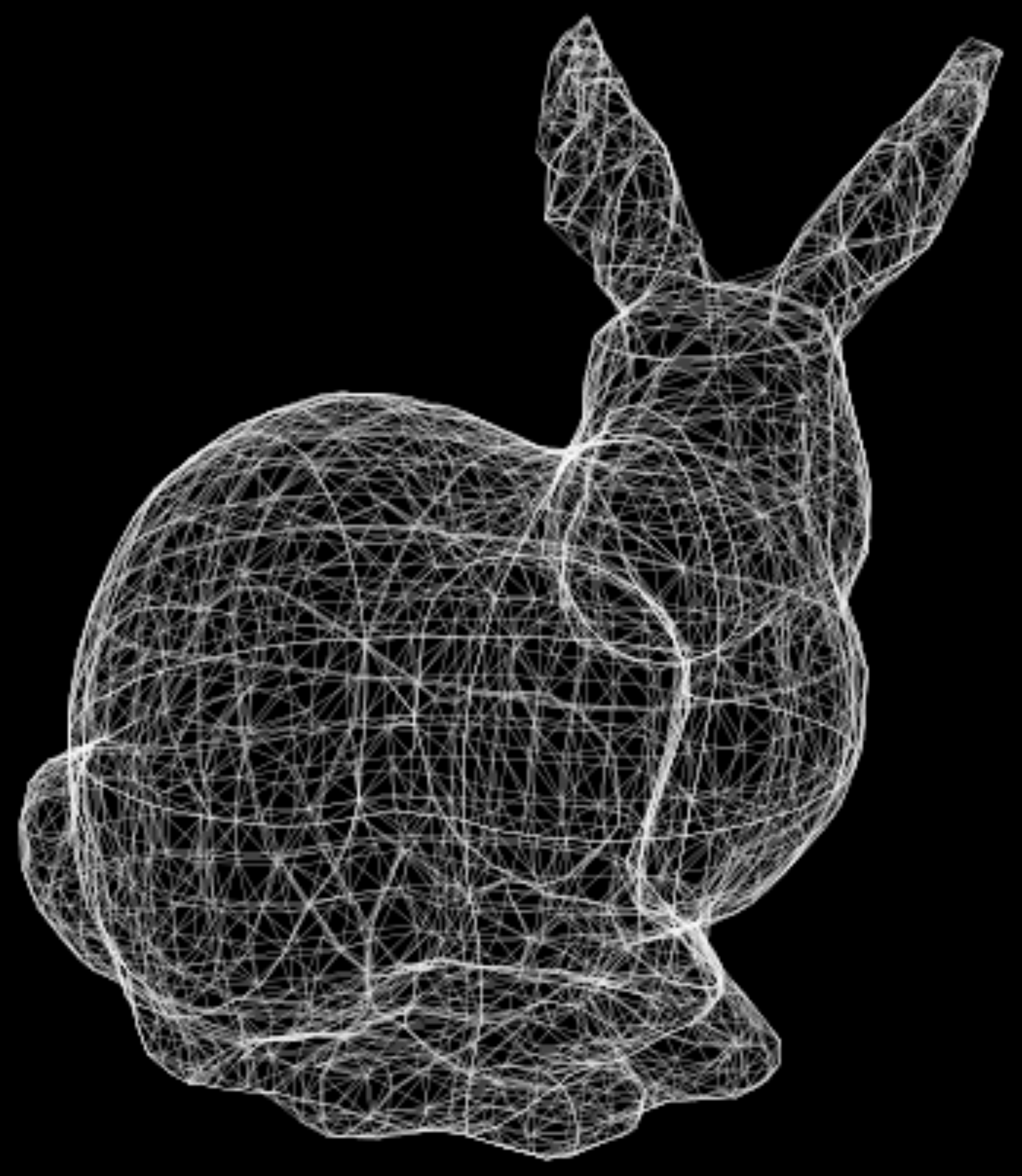
Discussions

- *Fast convergence*. Hash table entries are easy to optimize as learnable parameters.
- *Selective learning*. Eikonal constraint applies mainly near surfaces, focusing on a small subset of space.
- *Online adaptivity*. Allocations of hash table entries concentrate on a small region, and finer grids will experience fewer collisions while hashing (Müller et al., 2022).

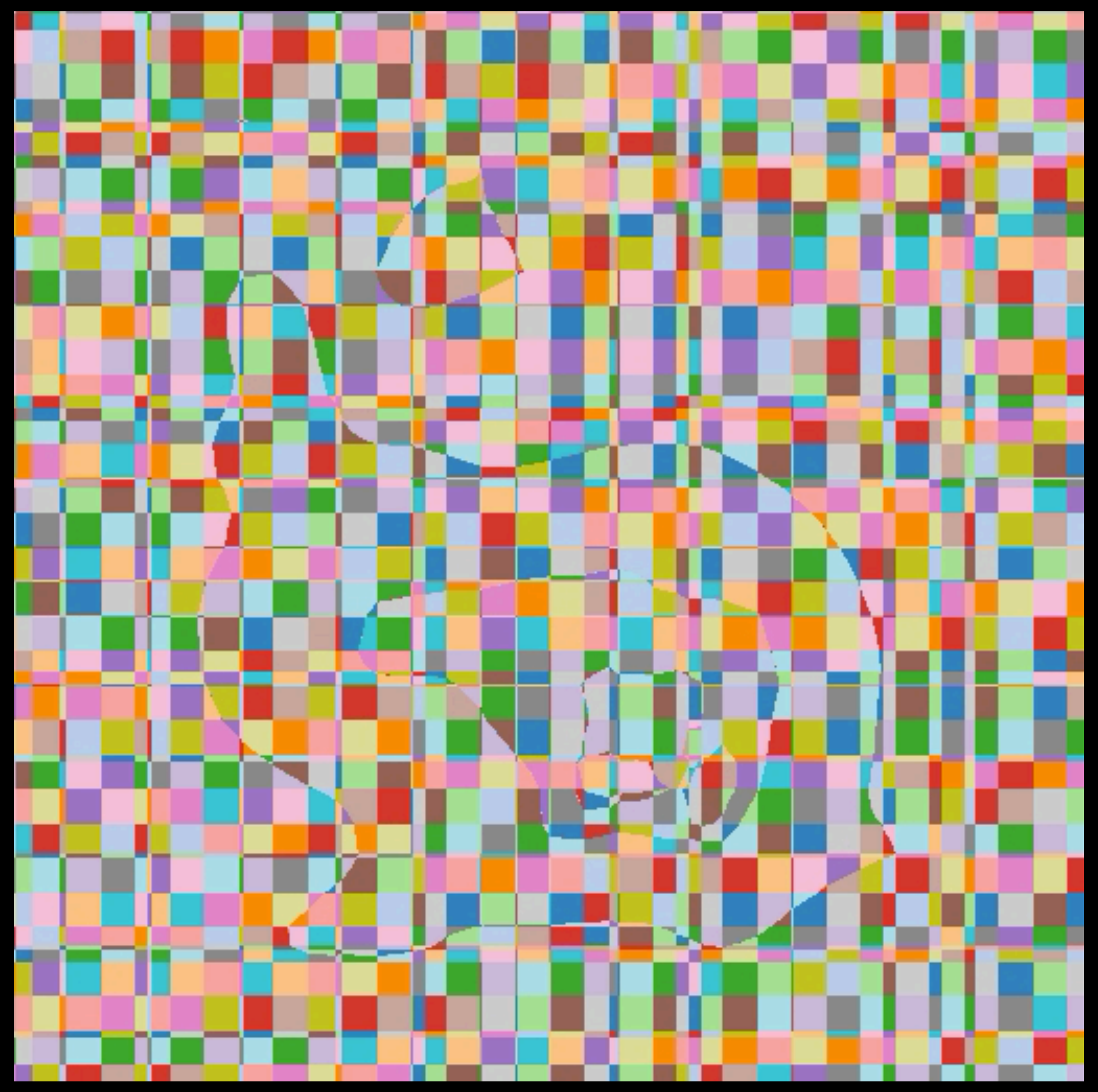
Visualizations



Normal map

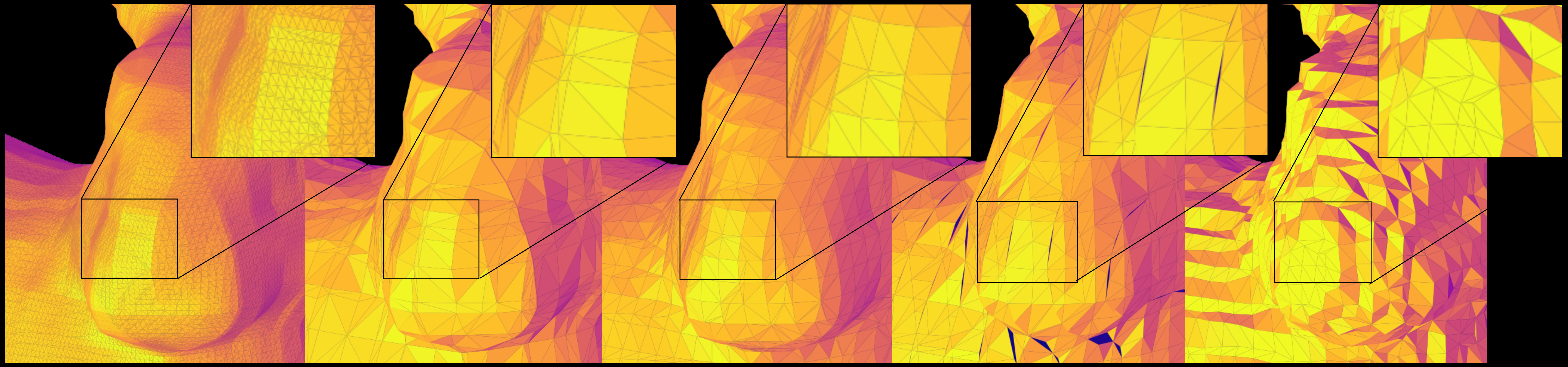


Skeleton



Trilinear regions shifting along the z-axis

Nose-to-nose comparison



MC 256
(92K Vertices as GT)

Ours
(4.5K Vertices)

MC 64
(5.6K Vertices)

MT 32
(4.7K Vertices)

NDC 64
(5.6K Vertices)

MT (Marching Tetrahedra), NDC (Neural Dual Contour; Chen et al., 2022)

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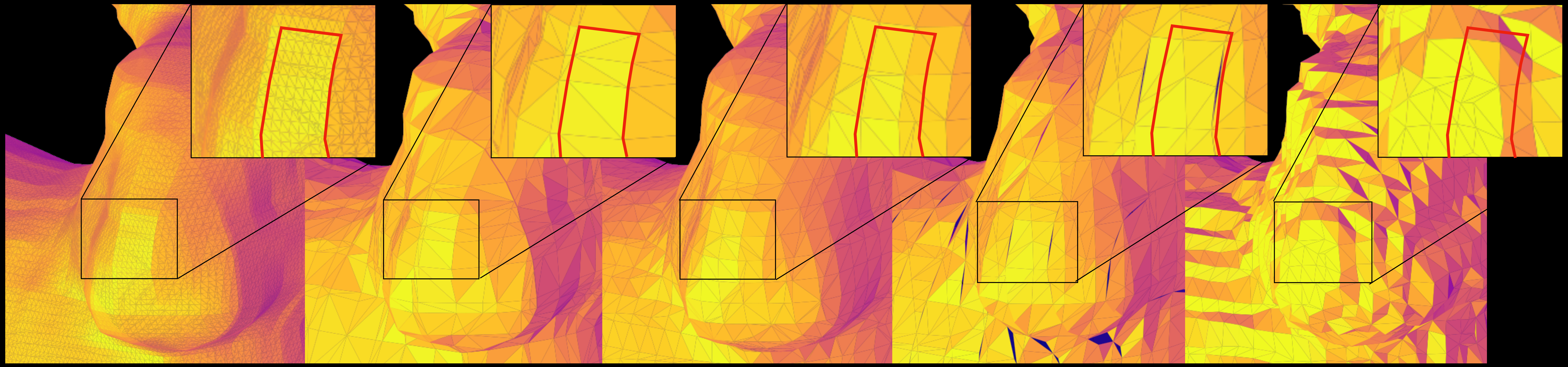

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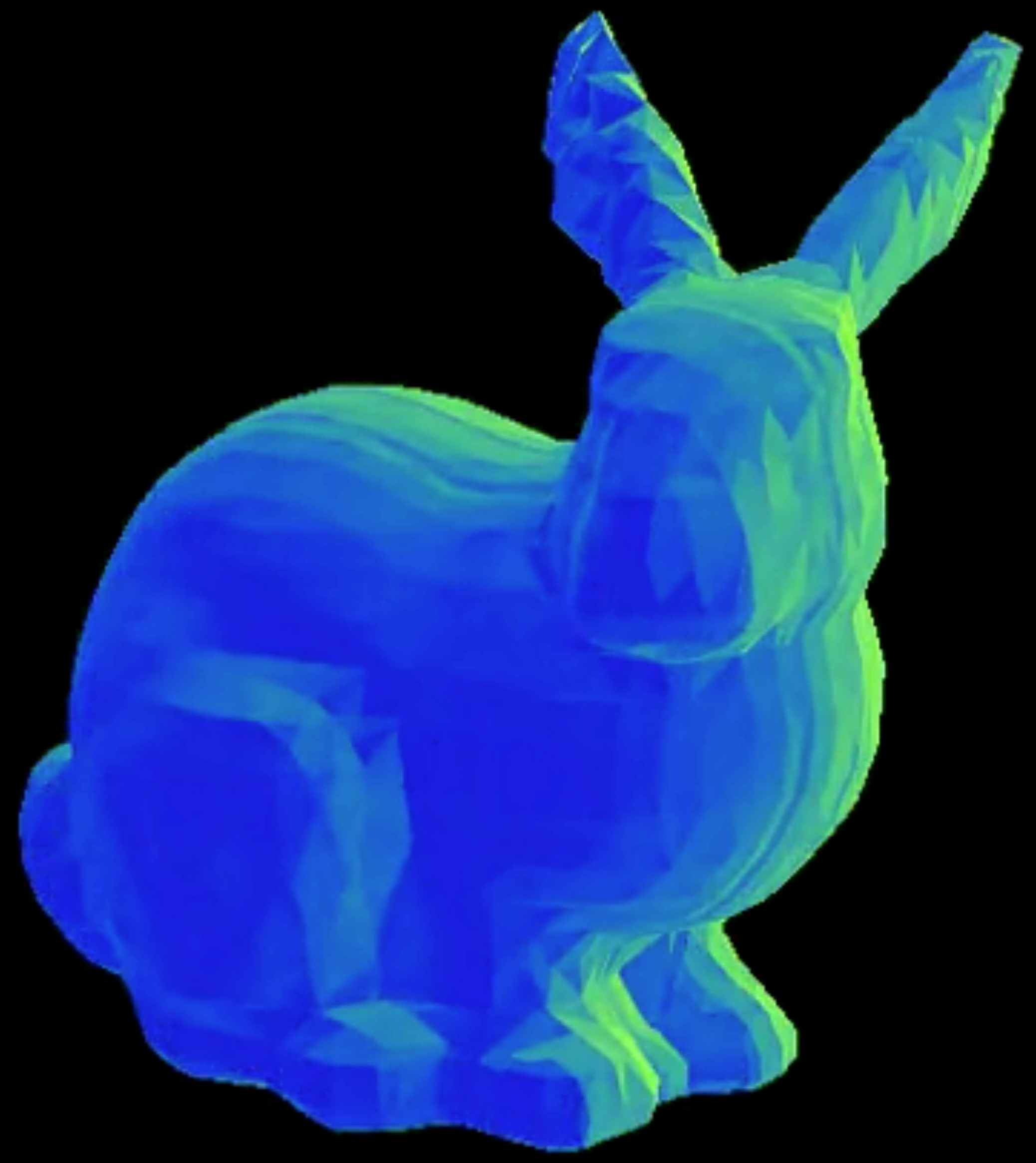
Deep dive into our paper

- Algorithms describing the whole procedure (Alg. 1 & 2)
- An approximation to get the intersection of three curved hypersurfaces (Thm. 4.7)
- Quantitative results on the Stanford 3D Scanning repository (Curless & Levoy, 1996) report the Chamfer distance and efficiency, the angular distance, and the time spent.
- The publicly available code[1] allows you to review implementation details for batch computations optimized for maximum parallelization.

[1] https://github.com/naver-ai/tropical-nerf.pytorch

Conclusions

- We present novel theoretical insights and a practical methodology for precise mesh extraction, employing *piecewise trilinear networks*.
- This provides *a theoretical exposition of the eikonal constraint*, revealing that within the trilinear region, the hypersurface transforms into a plane.
- We hope this novel discovery will inspire future work that explores innovative applications and further advancements in the field.



Thank you all!

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