# Learning Neural Contracting Dynamics: Extended Linearization and Global Guarantees

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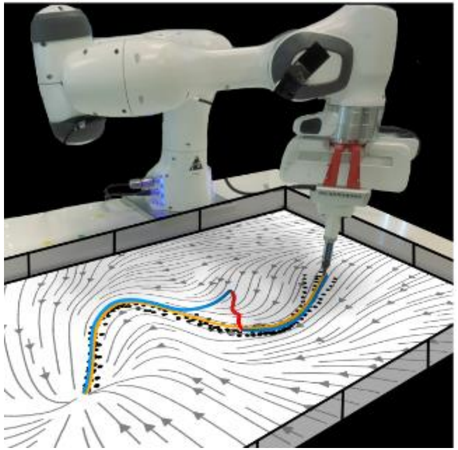
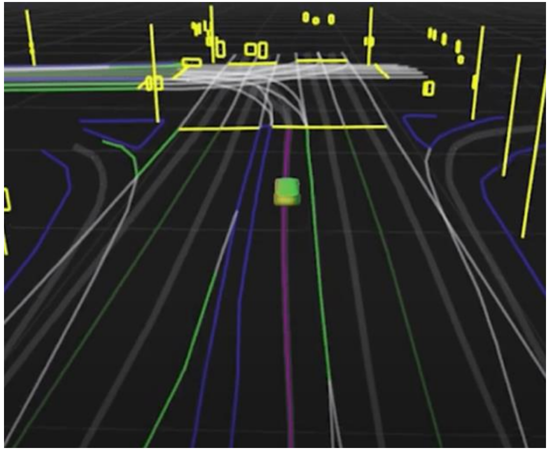
NEURAL INFORMATION PROCESSING SYSTEMS

# Imitation Learning





[1]H. Beik-Mohammadi, S. Hauberg, G. Arvanitidis, N. Figueroa, G. Neumann, and L. Rozo. Neural contractive dynamical systems. ICLR, 2024.
[2]https://neptune.ai/blog/self-driving-cars-with-convolutional-neural-networks-cnn

# Learning Dynamical Systems

Learn a System:

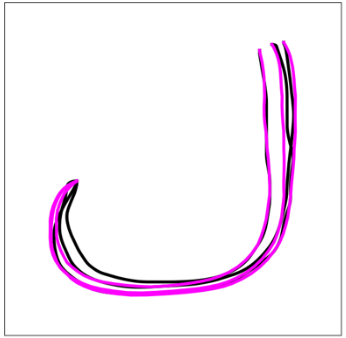
$$\{\{x_{i,t}, \dot{x}_{i,t}\}_{t=1}^{T}\}_{i=1}^{N}$$

From Demonstrations:

$$\dot{x} = f(x)$$

Which Most Closely Follows the Demonstrations:

$$\text{argmin}_f \sum_{i=1}^{N} \sum_{t=1}^{T} ||\dot{x}_{i,t} - f(x_{i,t})||_2$$



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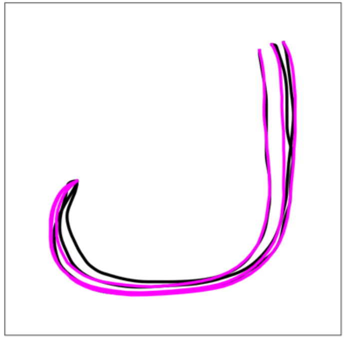


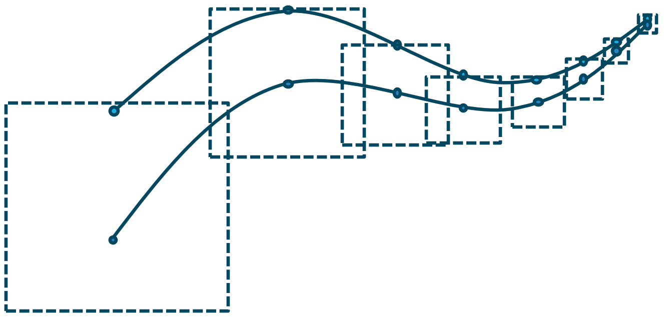
Which Most Closely Follows the Demonstrations:

$$\text{argmin}_f \sum_{i=1}^{N} \sum_{t=1}^{T} ||\dot{x}_{i,t} - f(x_{i,t})||_2$$

- Trajectories should all have same endpoint.
- Off data trajectories should smoothly rejoin data trajectories.
- System should be robust to noise.

# Contracting Dynamical System

Given $\dot{x} = F(x, t)$, vector field F is contractive if its flow is a contraction map.



Trajectories of contracting systems:

- Exponentially converge to a fixed point.
- Exponentially converge to each other.
- Are robust to disturbances.

W. Lohmiller and J.-J. E. Slotine. On contraction analysis for non-linear systems. Automatica, 34(6):683–696, 1998

# Neural Contractive Dynamical Systems

Parameterizes the Jacobian of the dynamical system

$$J_f(x) = -J_\theta(x)^\top J_\theta(x) - I$$

Contracting in L2 by construction:

$$\lambda_{max}\left(\frac{J_f + J_f^\top}{2}\right) < 0$$

Line-integrate Jacobian to get the vector field:

$$f(x) = \dot{x}_0 + \int_0^1 J_f((1-t)x_0 + tx)(x - x_0)dt$$

H. Beik-Mohammadi, S. Hauberg, G. Arvanitidis, N. Figueroa, G. Neumann, and L. Rozo. Neu-ral contractive dynamical systems. ICLR, 2024.

# Extended Linearized Contractive Dynamics

ELCD model

$$f(x) = A(x, x^*)(x - x^*)$$

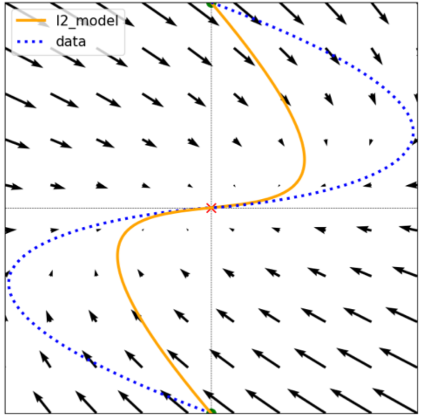
$$A(x) = -P_s(x, x^*)^\top P_s(x, x^*) + P_a(x, x^*) - P_a(x, x^*)^\top - \alpha I$$

- Parameterizes Vector Field, rather than Jacobian.
- Includes anti symmetric component.
- Contracting (in L2)

# Non-Euclidean Contracting Dynamics

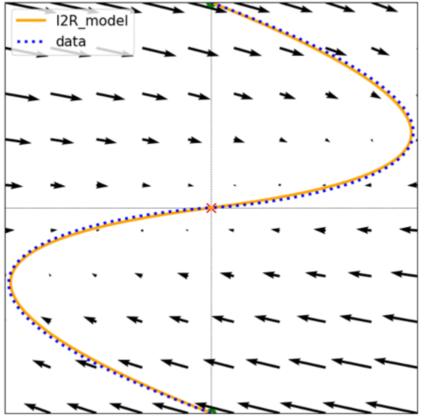
Use parameterization in latent space and use diffeomorphism to map to data space.

Without diffeomorphism




$$\dot{x} = A(x, x^*)(x - x^*),$$

With diffeomorphism




$$z = g(x)$$

Contraction is preserved under diffeomorphism, with respect to a different metric.

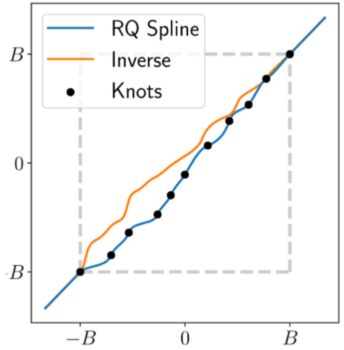
L. Dinh, J. Sohl-Dickstein, and S. Bengio. Density estimation using Real NVP. (ICLR), 2017.

# Choice in diffeomorphism

Coupling Layers[1]:

$$y_{1:d} = x_{1:d}$$
$$y_{d+1:D} = g_{x_{1:d}}(x_{d+1:D})$$

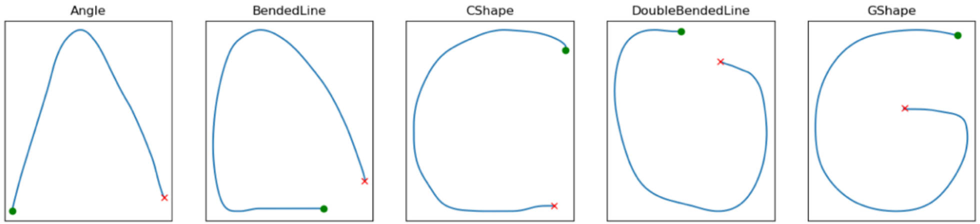
Equipped with Rational Quadratic Splines[2]:




[1]L. Dinh, J. Sohl-Dickstein, and S. Bengio. Density estimation using real NVP. ICLR, 2017
[2]C. Durkan, A. Bekasov, I. Murray, and G. Papamakarios. Neural spline flows. Neurips, 2019

# Datasets

1. LASA[1]:




2. n-link Pendulum:

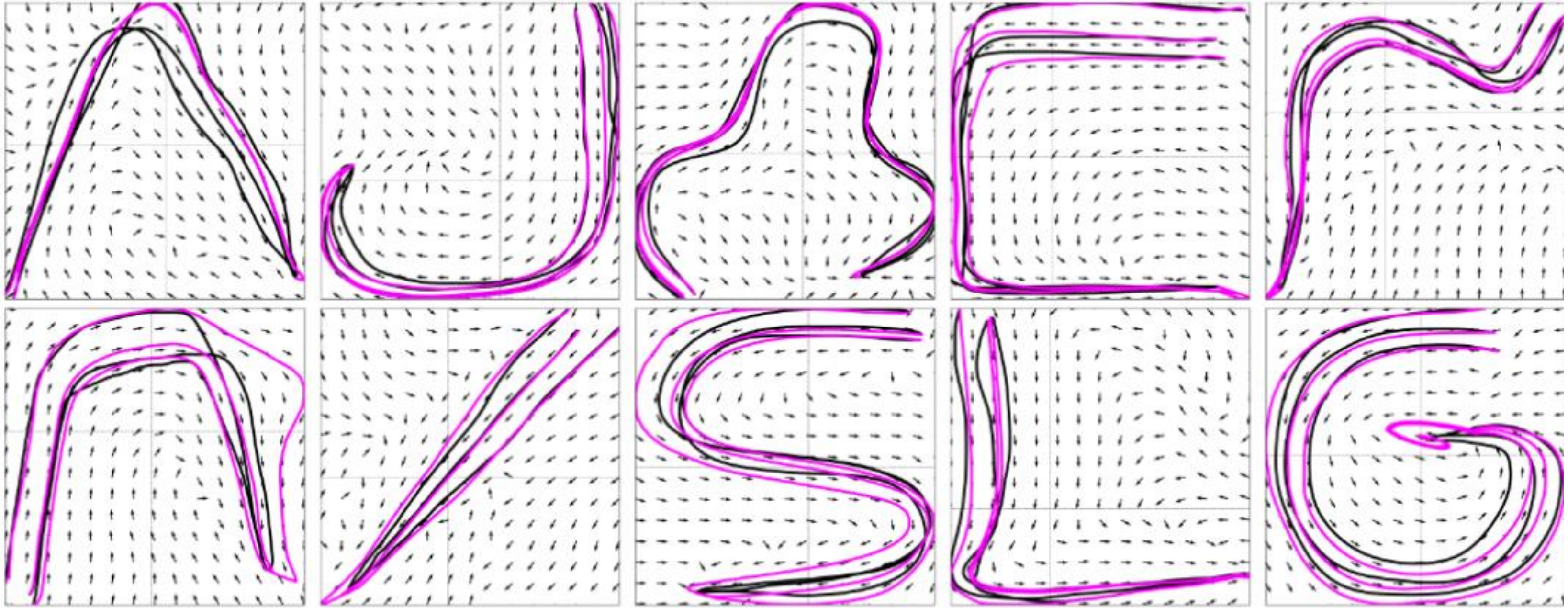
3. Rosenbrock:

$$f(x) = \lambda_1 (1 - x_1)^2 + \sum_{i=2}^{n} \lambda_i (x_i - x_{i-1}^2)^2$$

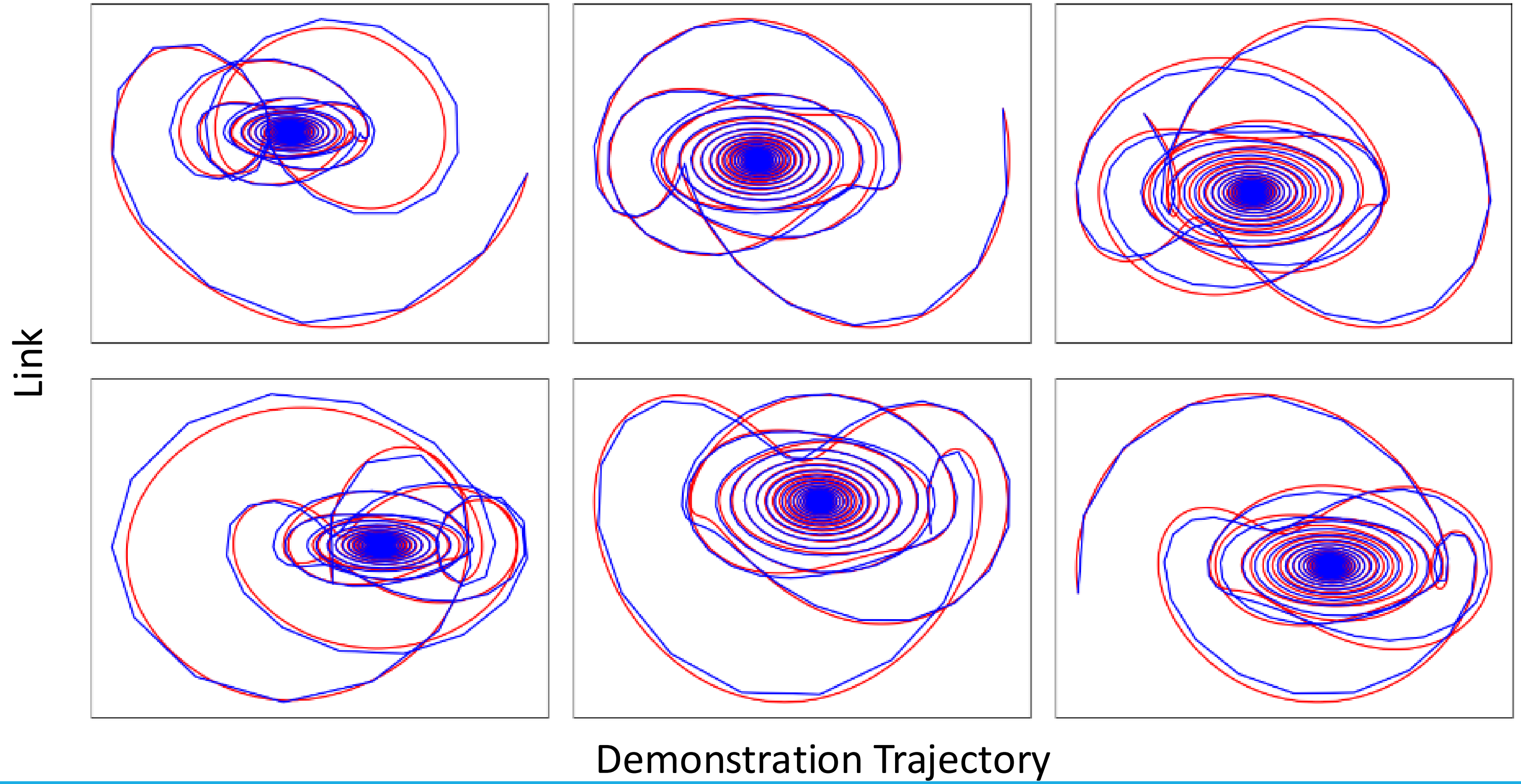
$$\dot{x} = -G(x)^{-1} \nabla f(x),$$

[1]A. Lemme, Y. Meirovitch, M. Khansari-Zadeh, T. Flash, A. Billard, and J. J. Steil. Open-source benchmarking for learned reaching motion generation in robotics. Paladyn, Journal of Behavioral Robotics, 6(1):30–41, 2015

# Results



# 2-Link Pendulum Phase Plots



# Results

| | SDD | EFlow | NCDS | ELCD |
|---|---|---|---|---|
| LASA-2D | $0.37 \pm 0.32$ | $1.05 \pm 0.25$ | $0.59 \pm 0.61$ | $\mathbf{0.12 \pm 0.11}$ |
| LASA-4D | $2.49 \pm 2.4$ | $2.24 \pm 0.12$ | $2.19 \pm 1.23$ | $\mathbf{0.80 \pm 0.54}$ |
| LASA-8D | $5.26 \pm 0.50$ | $2.66 \pm 0.63$ | $5.04 \pm 0.77$ | $\mathbf{1.52 \pm 0.61}$ |
| Pendulum-4D | $0.49 \pm 0.11$ | $0.17 \pm 0.01$ | $1.35 \pm 2.26$ | $\mathbf{0.03 \pm 0.01}$ |
| Pendulum-8D | $0.75 \pm 0.08$ | $0.33 \pm 0.01$ | $2.88 \pm 0.69$ | $\mathbf{0.14 \pm 0.03}$ |
| Pendulum-16D | $1.86 \pm 0.14$ | $0.45 \pm 0.01$ | $1.65 \pm 0.31$ | $\mathbf{0.44 \pm 0.09}$ |
| Rosenbrock-8D | NaN | $1.90 \pm 0.16$ | $2.74 \pm 0.15$ | $\mathbf{1.22 \pm 0.01}$ |
| Rosenbrock-16D | NaN | $3.57 \pm 0.66$ | $3.68 \pm 0.12$ | $\mathbf{2.57 \pm 0.09}$ |

Mean and standard deviation of dynamic time warping distance (DTWD) of baselines and ELCD on all datasets.