# Can we count on Deep Learning: Characterizing Combinatorial Structures with Interpretability

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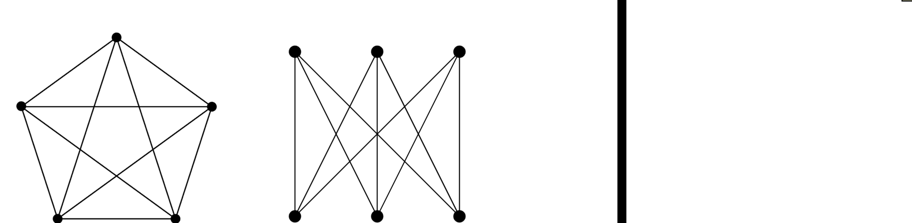
**Feature Attribution Clustering for Exploration (FACE) is a method to help mathematicians characterize sets of mathematical objects via prototypical feature attribution maps.**

**Examples of characterization problems:**

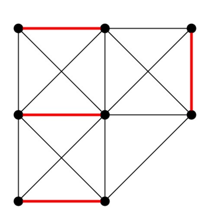
Q: *Which graphs are planar?*

A: Planar ⟷ no $K_5$, $K_{3,3}$ minors

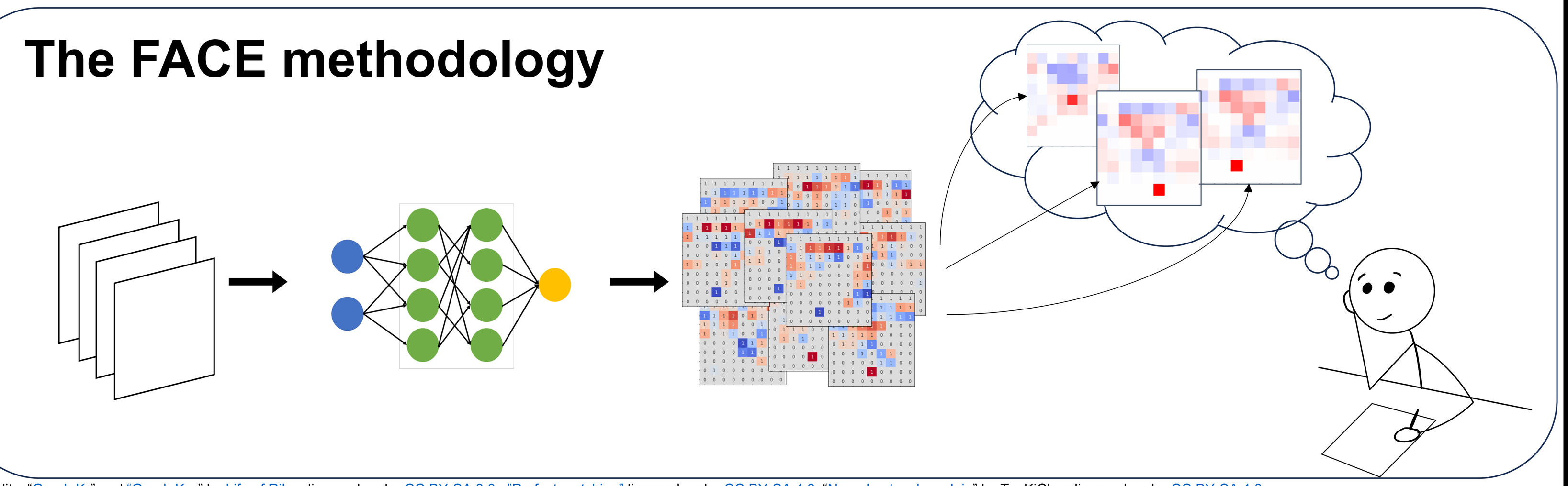


Q: *Which graphs have perfect matchings?*

A: $G$ has a perfect matching ⟷ for every subset of vertices $S$, $G \setminus S$ has at most $|S|$ odd components

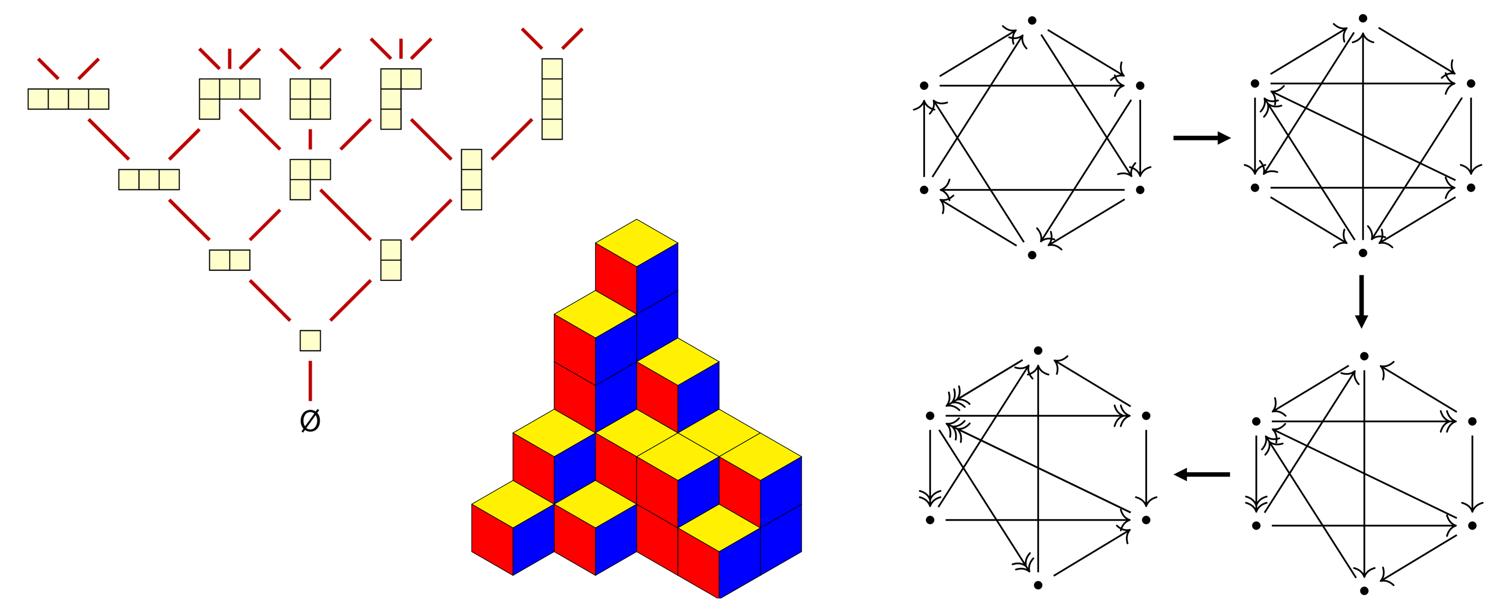


**The FACE methodology**

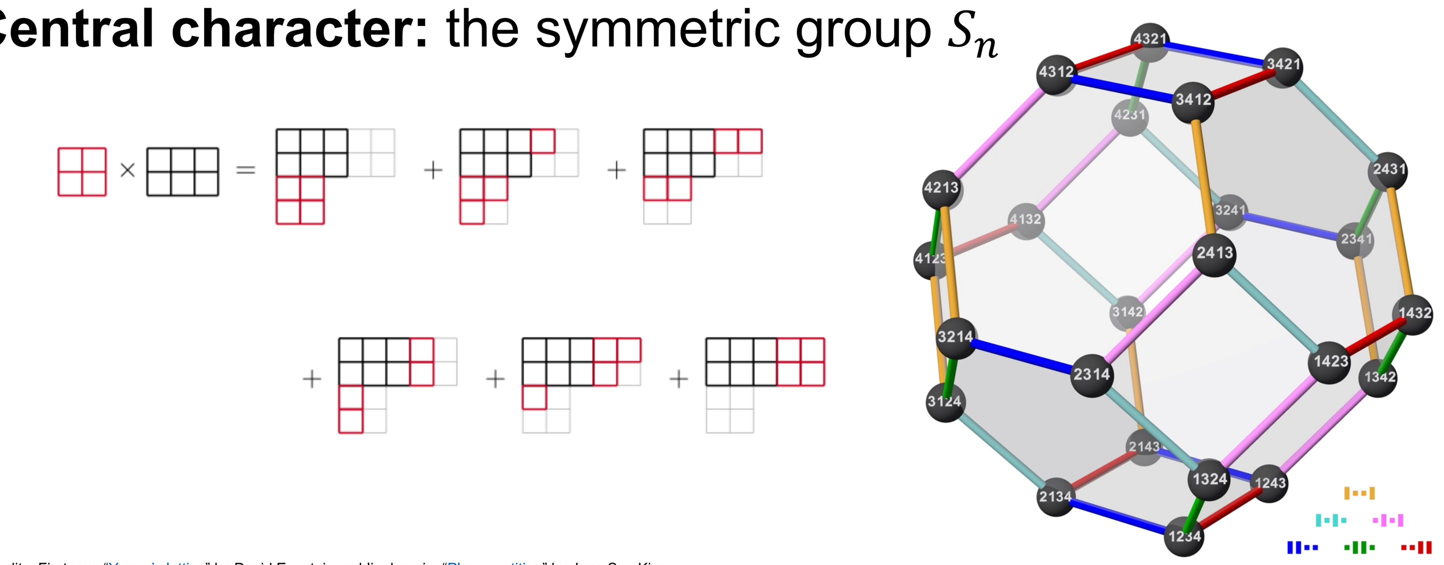




## Algebraic combinatorics



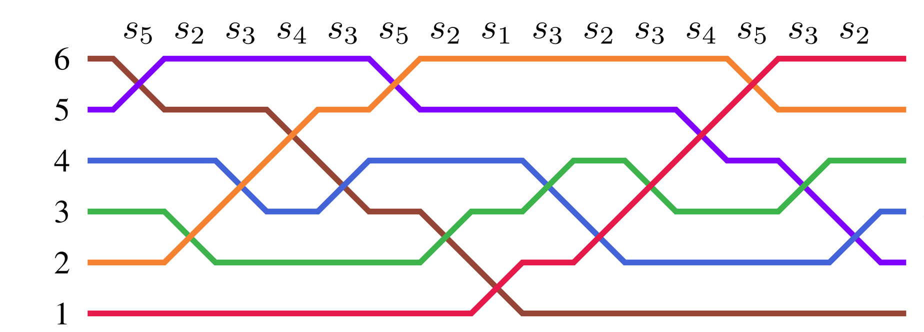
**Central character:** the symmetric group $S_n$





## Weaving patterns

Any permutation can be written as a product of adjacent transpositions, called a *reduced word.*



| | | | | | |
|---|---|---|---|---|---|
| 6 | 1 | 1 | 1 | 1 | 1 |
| 5 | 0 | 1 | 1 | 1 | 1 |
| 4 | 1 | 0 | 1 | 1 | 0 |
| 3 | 1 | 0 | 0 | 1 | 0 |
| 2 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 |

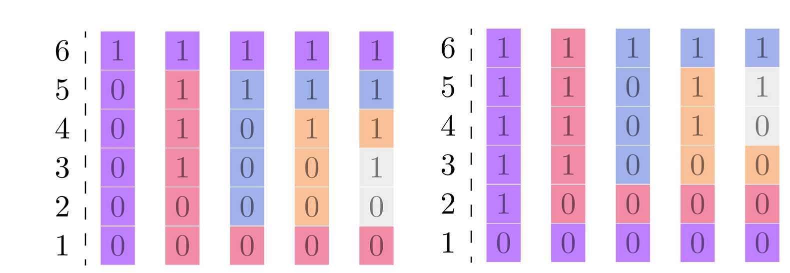
Every reduced word that differs by a *commutation relation* has the same weaving pattern, e.g. $s_1s_2s_3s_1s_2s_1 \sim s_1s_2s_1s_3s_2s_1$

Q: Which {0,1} matrices are weaving patterns?

Some families of reduced words have weaving patterns with nice characterizations.

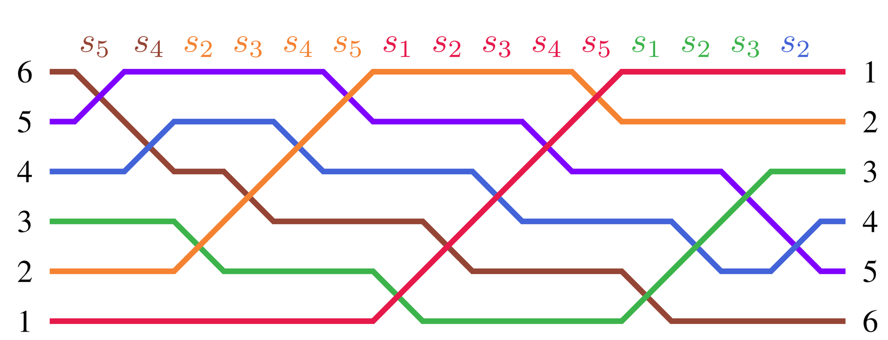
| | | | | | |
|---|---|---|---|---|---|
| 6 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 0 | 1 | 1 | 1 |
| 4 | 0 | 1 | 1 | 1 | 0 |
| 3 | 1 | 0 | 0 | 1 | 0 |
| 2 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 |

{0, 1} matrix that is not a weaving pattern



Examples of weaving patterns of *ordered words*

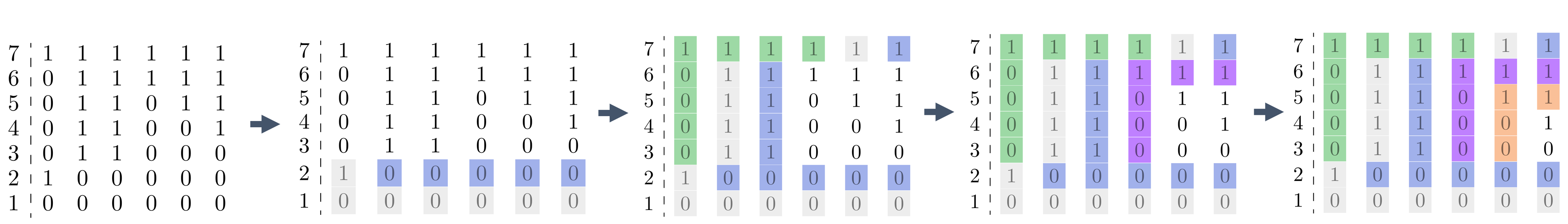
## Case Study: Two-Sided Ordered Words



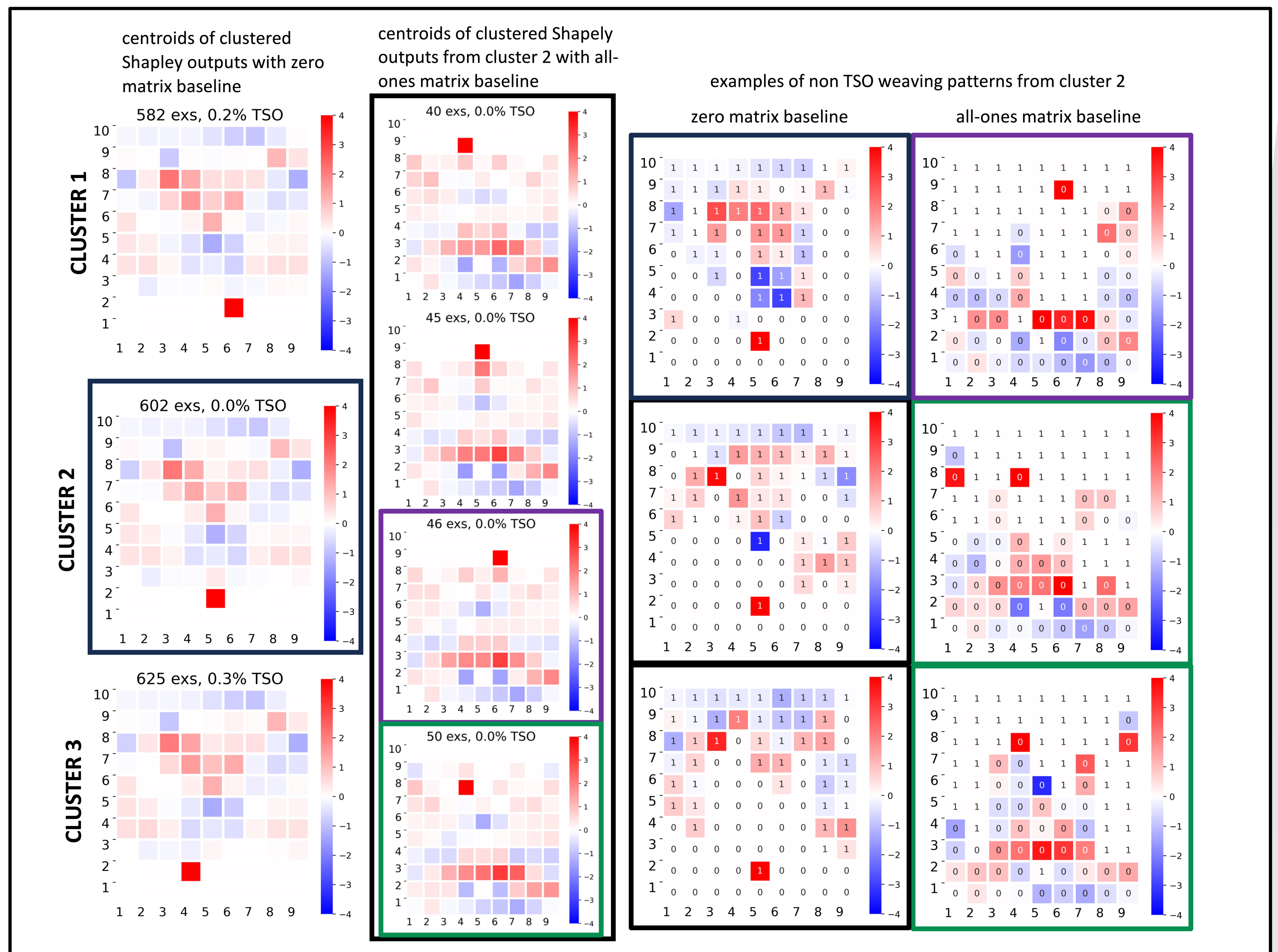
| | | | | | |
|---|---|---|---|---|---|
| 6 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 0 |
| 4 | 0 | 0 | 1 | 1 | 1 |
| 3 | 1 | 0 | 0 | 0 | 1 |
| 2 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 |

### Application of FACE

- **Create dataset:** Generated all TSO weaving patterns for $n = 9$ (48,896 $10 \times 9$ {0,1} arrays) and the same number of non-TSO weaving patterns.
- **Train model:** Trained a CNN with two convolutional layers; 99% test accuracy
- **Calculate feature attribution representations:** Applied Shapley to 16,000 weaving patterns from the test set.
- **Find prototype feature attributions through clustering:** Clustered Shapley outputs using $k$-means and calculated centroids for each cluster.
- **Mathematician analysis of prototypes**



Example of application of algorithm to determine whether weaving pattern is TSO

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