Beyond Geometry: Comparing the Temporal Structure of Neural Circuits with **Dynamical Similarity Analysis (DSA)**

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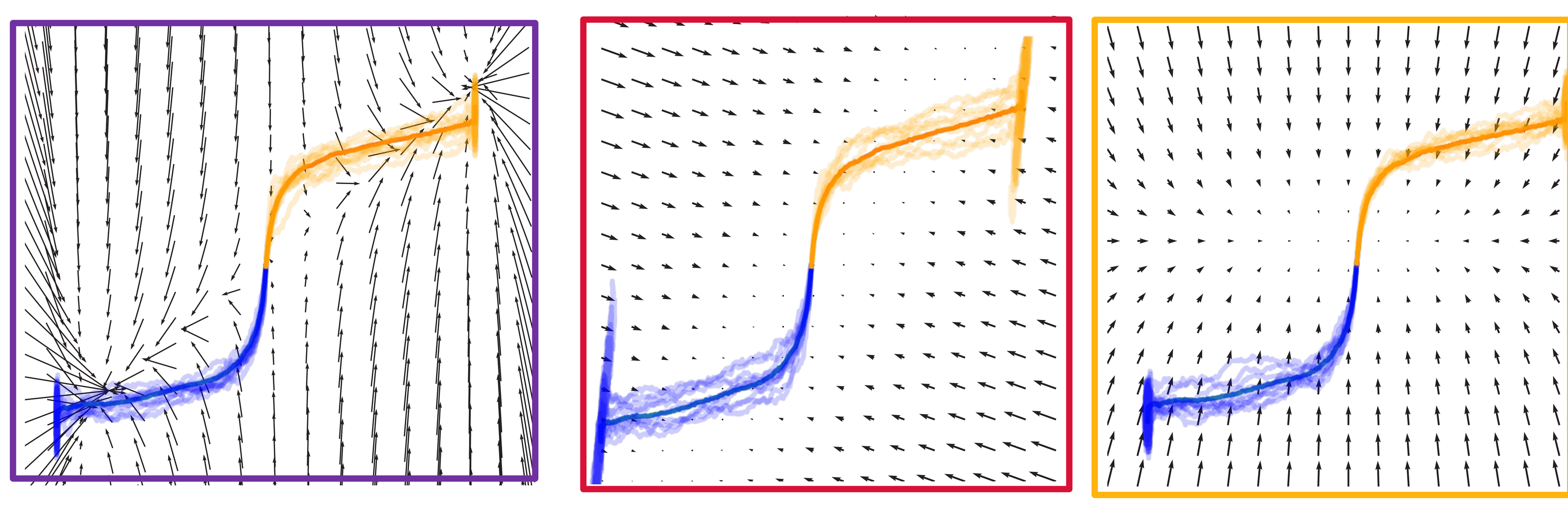
Highlights

- Dynamics and geometry are distinct levels for neural systems.
- Dynamics describe the core mechanisms of neural computation.
- Current methods for comparing neural networks are purely geometric.
- Our novel method, **DSA**, identifies dynamical similarities + differences between two systems.
- We leverage delay embeddings and Koopman operator theory to create a data-driven comparison method that can disentangle geometry + dynamics.

Dynamics ≠ Geometry[1]

Bistable switch | Line Attractor | Point Attractor

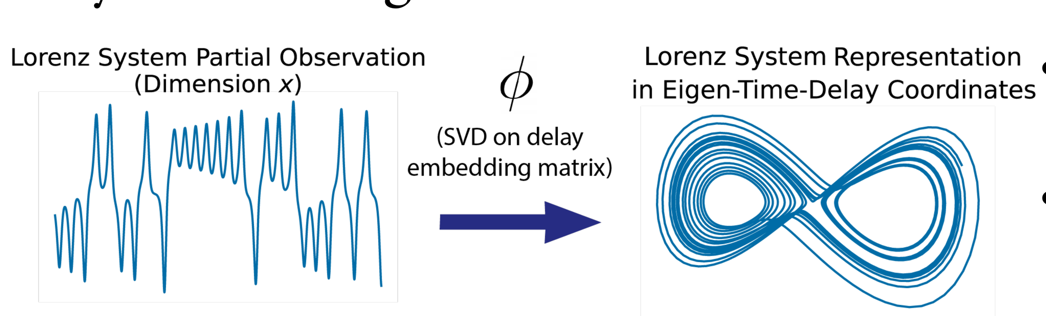


Shape Metrics

BrainScore (Linear Regression + Pearson Correlation)[2]	$\frac{\text{cov}(Y, WX)}{\sqrt{[\text{var}(Y)\text{var}(Wx)]}}$
Centered Kernel Alignment[3]	$\frac{X^T Y}{\lvert X\rvert_2 \lvert Y\rvert_2}$
Procrustes Analysis[4]	$\min_{C \in O(n)} \lvert X - CY\rvert_F$

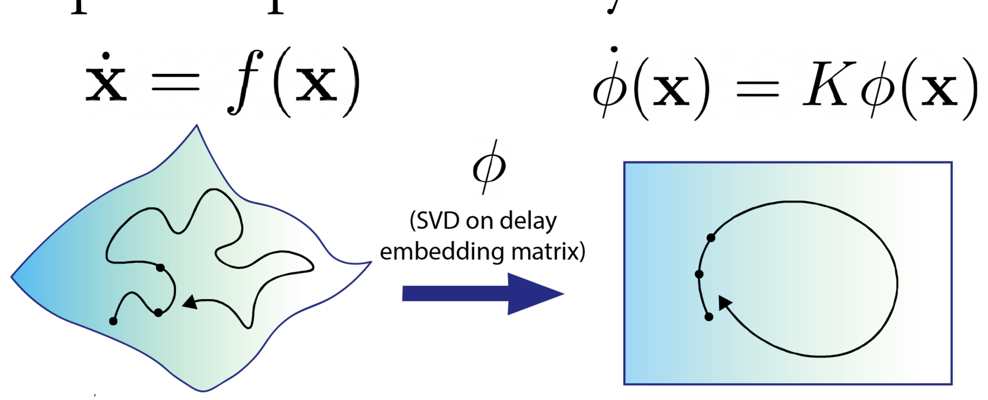
Theoretical Background

Delay Embeddings



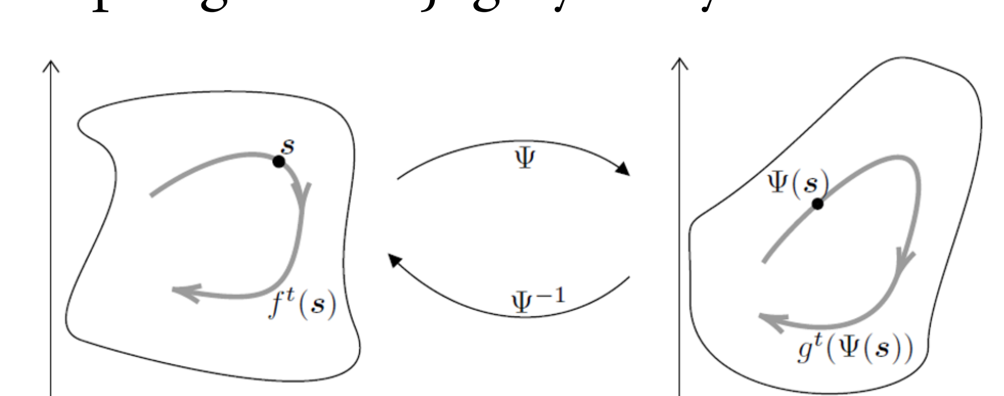
- Efficient method for nonlinear embedding.
- Allows us to reconstruct partially-observed systems.[5]

Koopman Operator Theory



- A global linear description of a nonlinear system achieved by embedding observations into a Hilbert Space.[6]
- Finite approximation: Dynamic Mode Decomposition (HAVOK[7]).

Topological Conjugacy of Dynamics



$$g^t \circ \Psi(s) = \Psi \circ f^t(s)$$

- Conjugate systems have the same dynamical features: fixed points, limit cycles, invariant manifolds, etc.
- i.e. their shapes may be different, but they're doing the same thing.

Problem: Shape Metrics only measure Geometric Similarity!

Our Solution: **DSA**

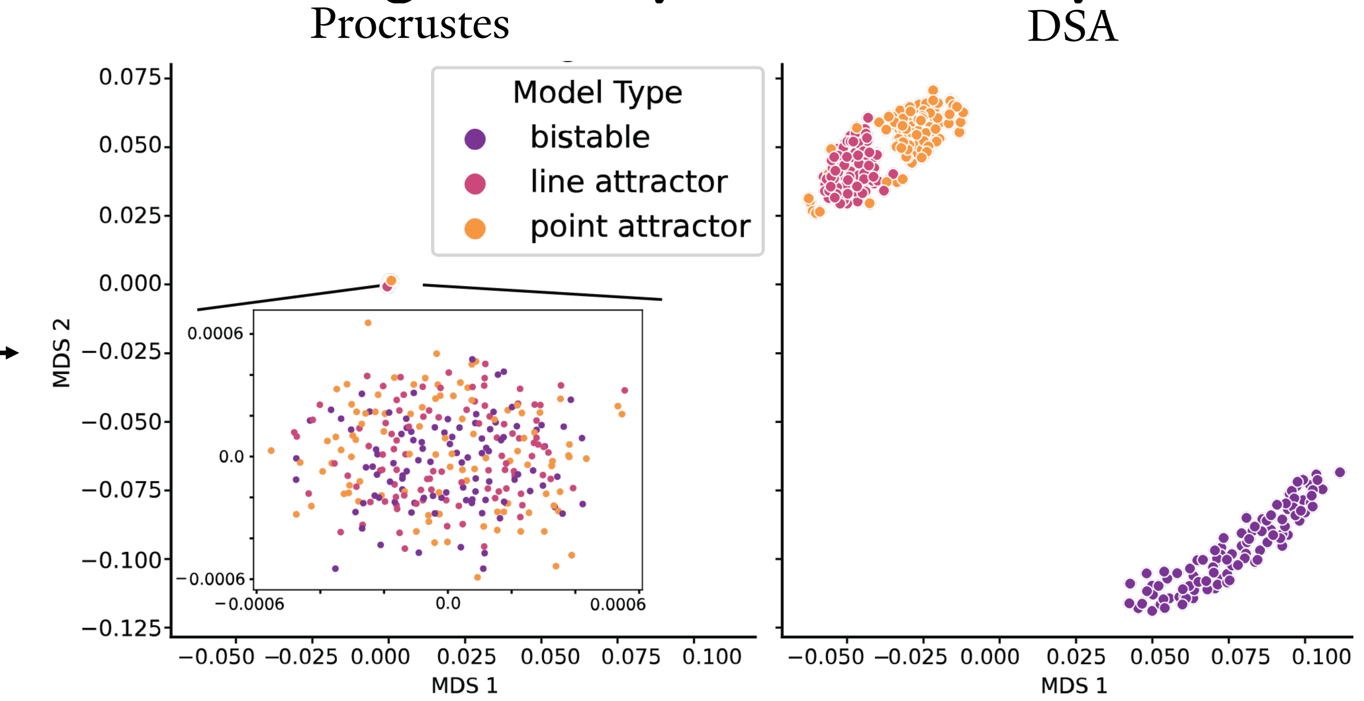
1. Sample time-series data from two dynamical systems: $\{x_t, x_{t+1}, \ldots\}$, $\{y_t, y_{t+1}, \ldots\}$
2. Nonlinearly embed these into a higher dimension: Φ_1: $\{\tilde{x}_t, \tilde{x}_{t+1}, \ldots\}$; Φ_2: $\{\tilde{y}_t, \tilde{y}_{t+1}, \ldots\}$
3. Fit a linear model to the data (next-step prediction): $\tilde{x}_{t+1} = A_x \tilde{x}_t$, $\tilde{y}_{t+1} = A_y \tilde{y}_t$
4. Compare dynamics matrices

$$DSA(A_x, A_y) = \min_{C \in O(n)} \lvert A_x - C A_y C^{-1} \rvert_F$$

Properties

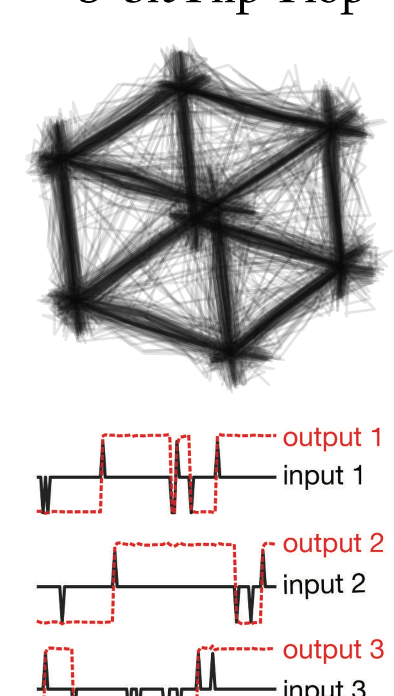
1. *DSA is a proper metric.*
2. *Metric properties can be relaxed to purely identify conjugacy (by optimizing C over GL(n).*
3. *DSA is equivalent to the 2-Wasserstein Distance over the eigenvalues of A_x, A_y, when the dynamics matrices are normal.*
4. *We optimize DSA using gradient descent over the Cayley Transform of arbitrary matrices.*

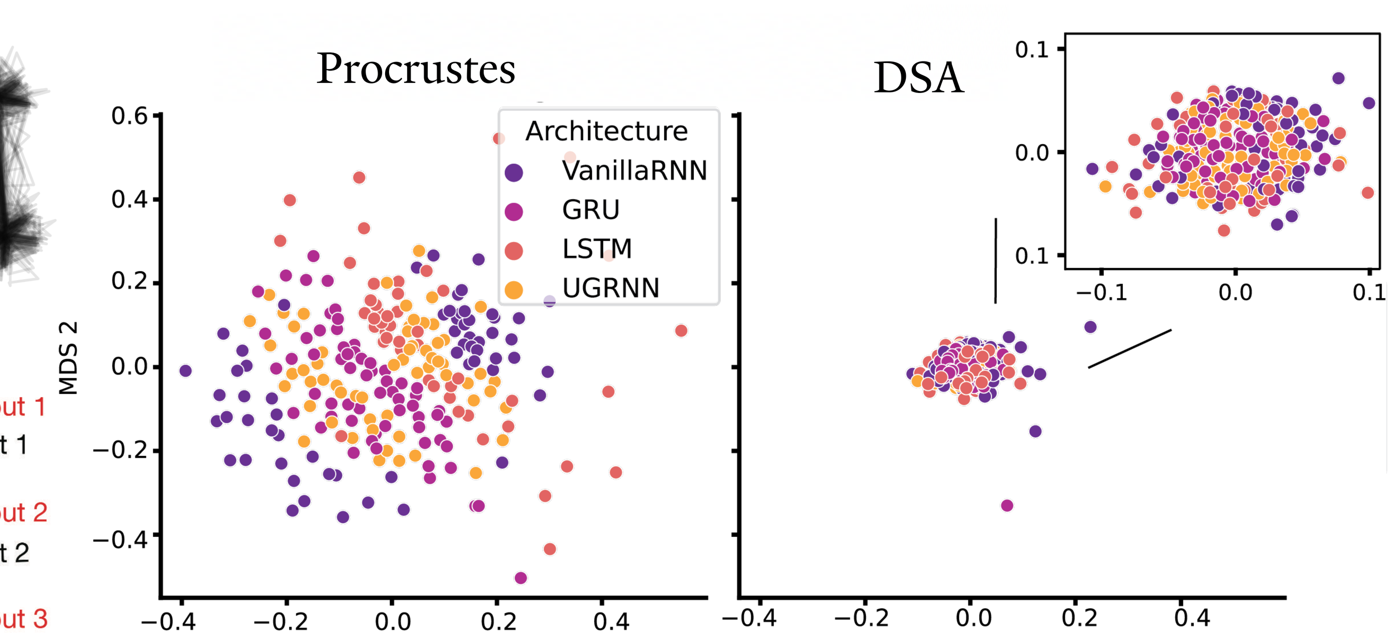
Results: Same geometry, different dynamics



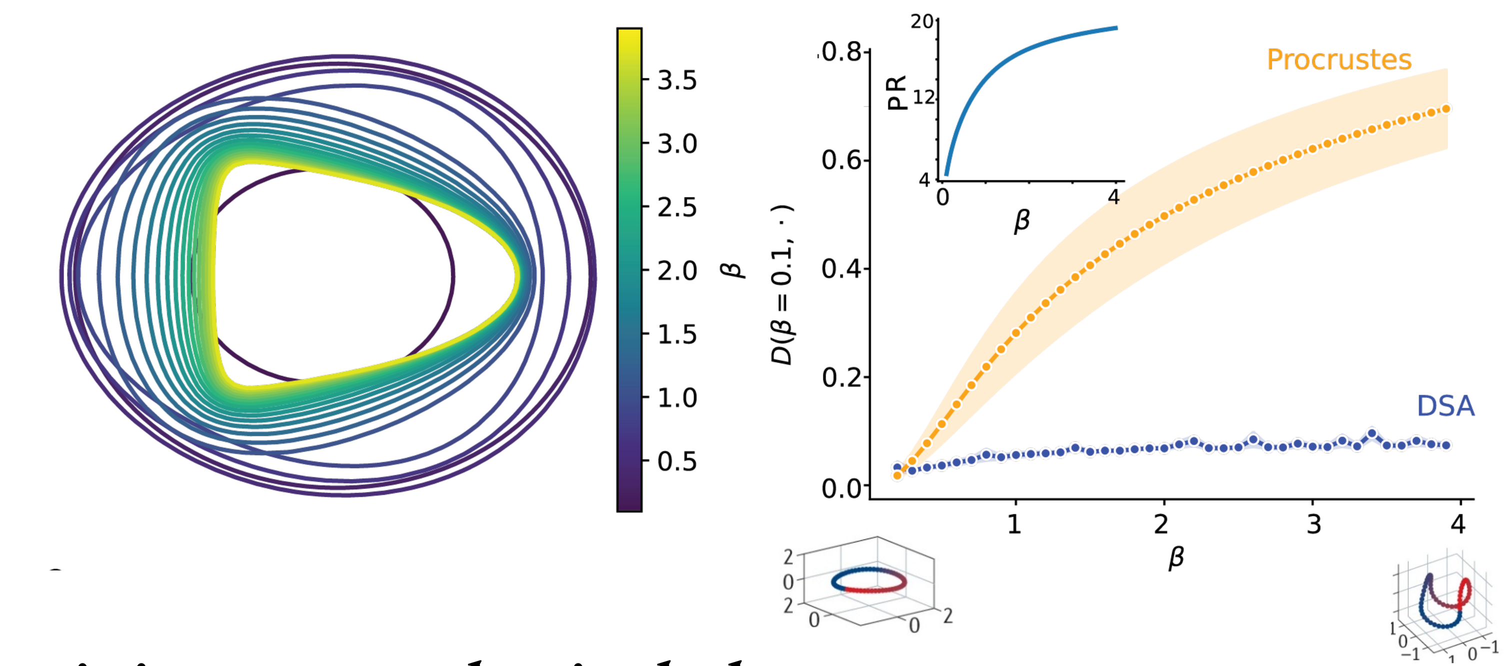
Same dynamics, different geometry

3-bit Flip-Flop[8]

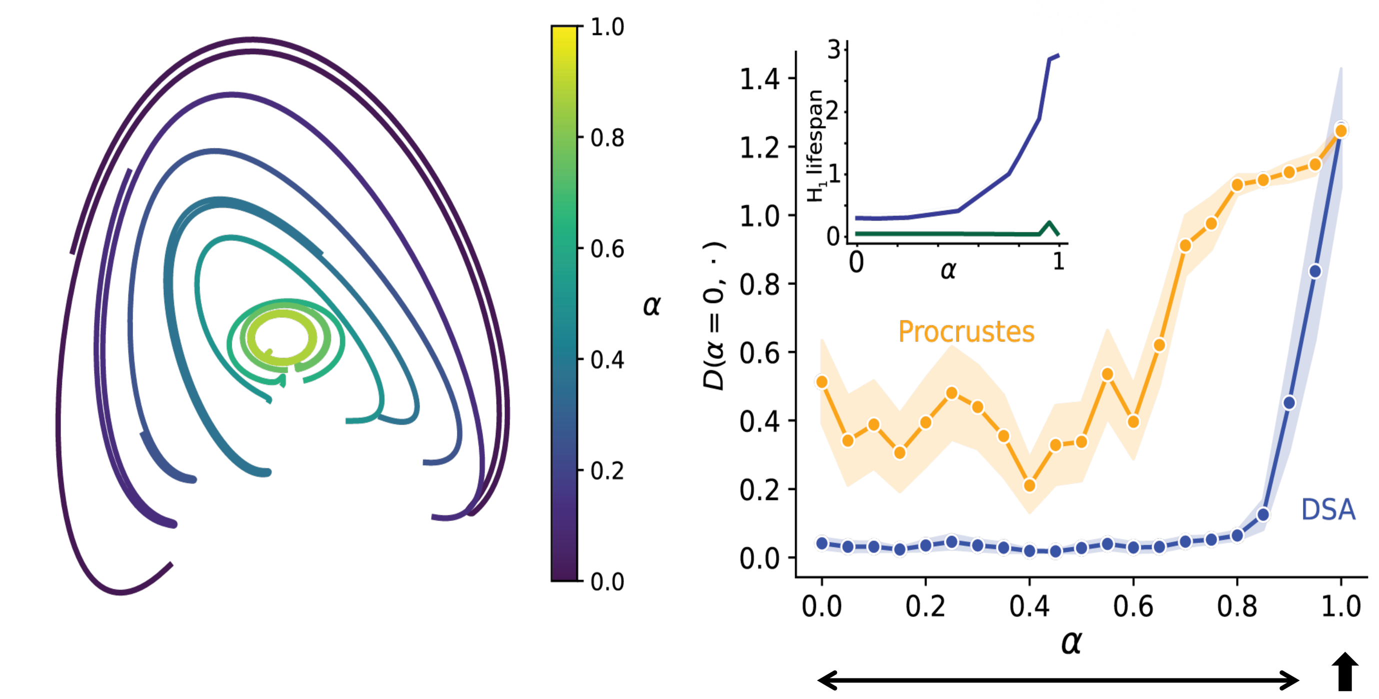




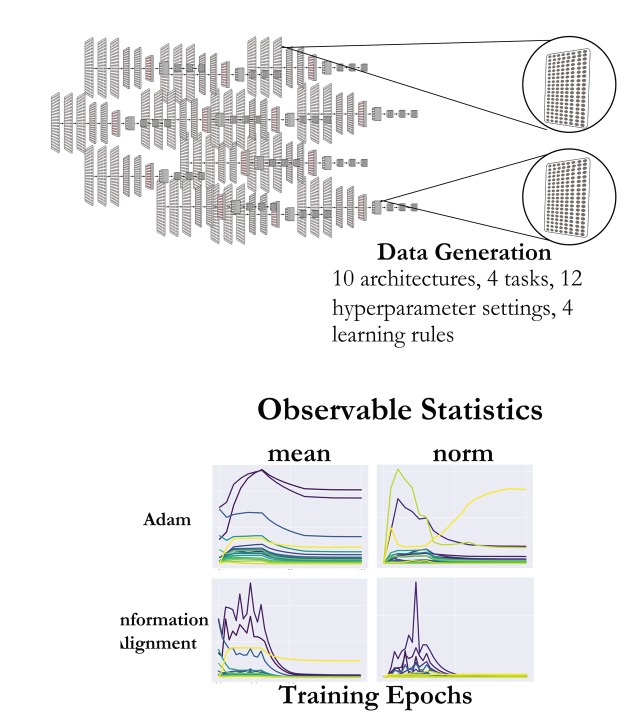
Rings: Invariance to non-topological deformation

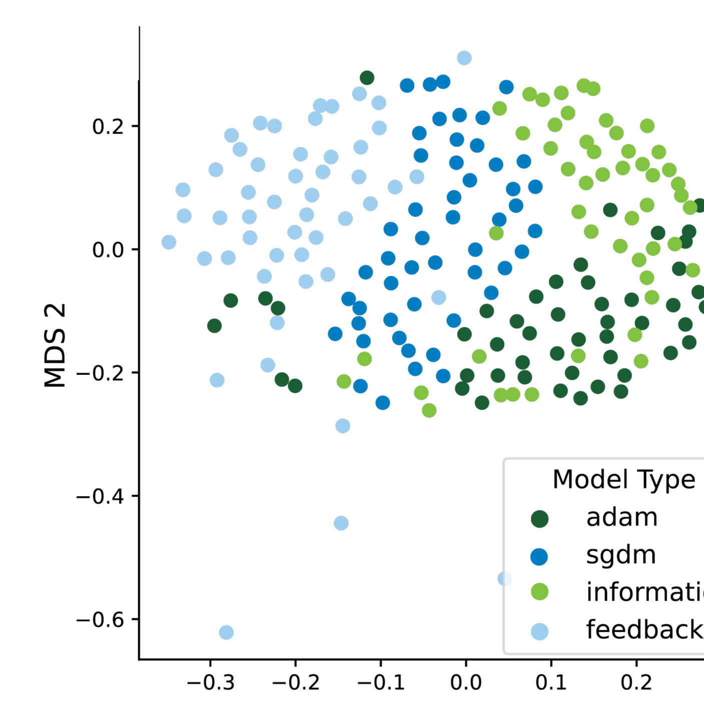


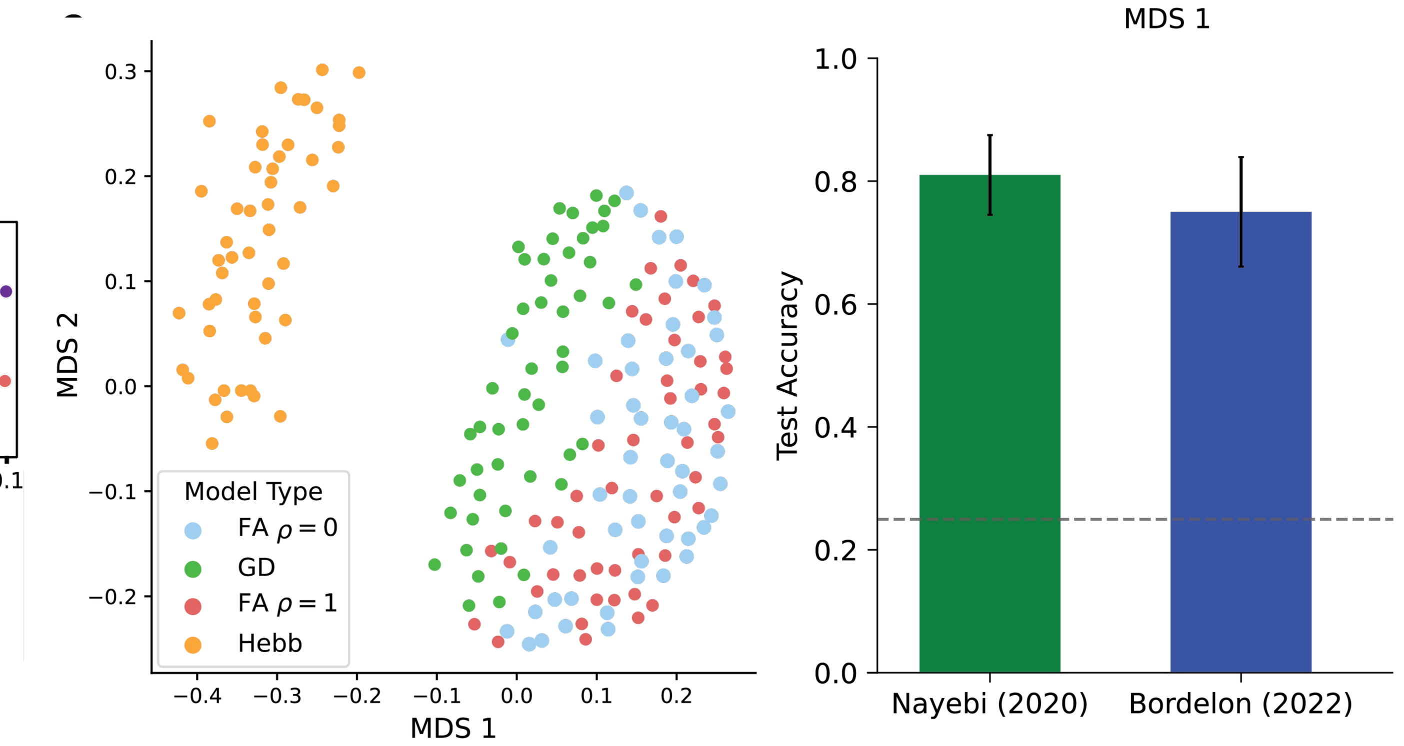
Selectivity to topological changes



Application: Identifying learning rules[9,10]







[1] Galgali 2023 [2] Schrimpf 2018 [3] Kornblith 2019 [4] Williams 2021 [5] Takens 1981 [6] Koopman 1931 [7] Brunton 2017 [8] Maheswaranathan 2019 [9] Nayebi 2021 [10] Bordelon and Pehlevan [2022]