Koopman Kernel Regression

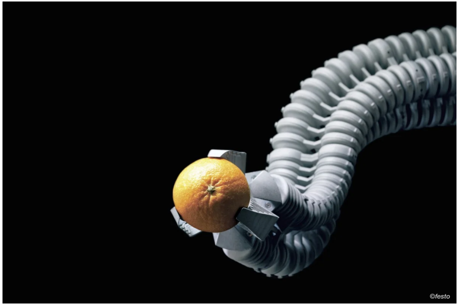
Petar Bevanda[1] Max Beier[1] Armin Lederer[1] Stefan Sosnowski[1]
Eyke Hüllermeier[2] Sandra Hirche[1]

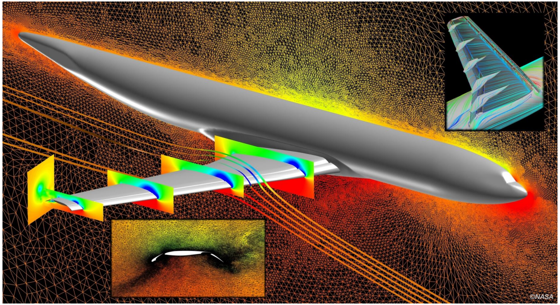
[1]Chair of Information-oriented Control
Technical University of Munich

[2]Chair of Artificial Intelligence and Machine Learning
Ludwig Maximilian University of Munich

37th Conference on Neural Information Processing Systems (NeurIPS 2023)

Simple yet Expressive Representations of Complex Dynamics





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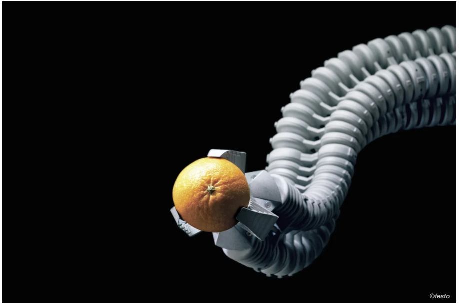

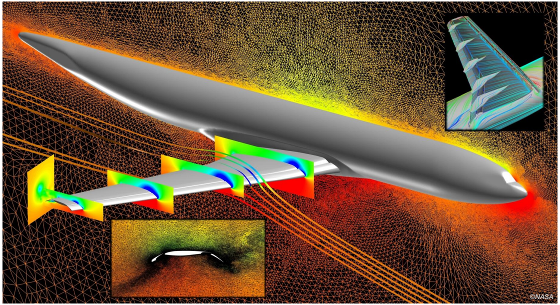

- **simple** & universal **representations**

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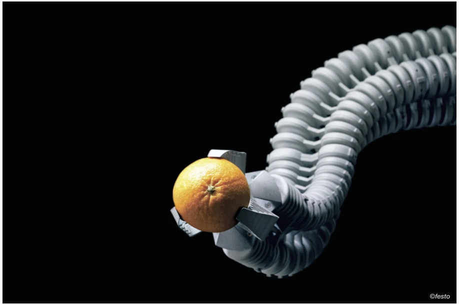

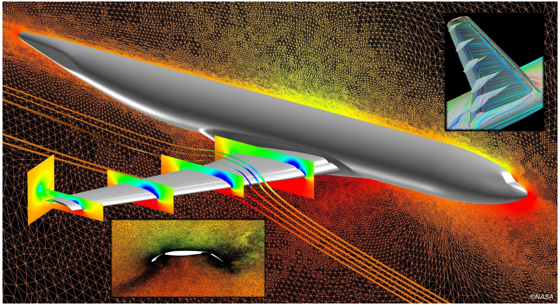

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- guaranteed **generalization** & **consistency**

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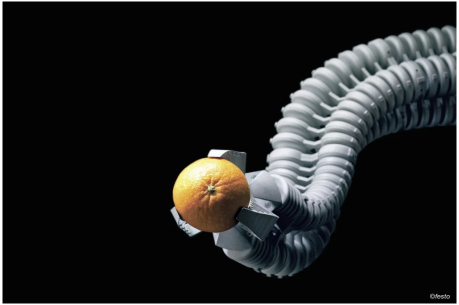

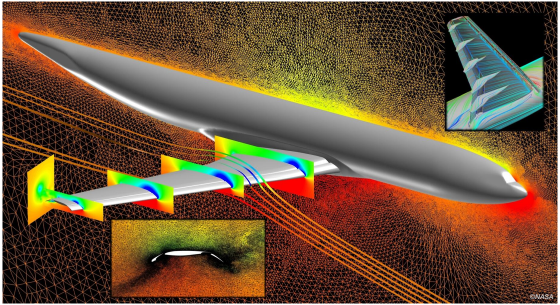

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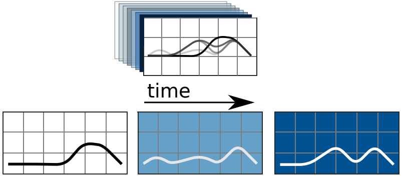
$\Rightarrow$ *facilitating accelerated optimization-based decision making*

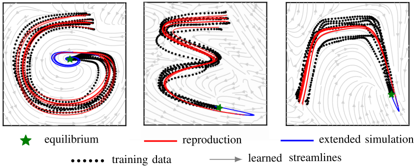
Take the best of time-series, ODEs and RKHS

Time-series decomposition



- discriminative ✓
- linear ✓
- time-variant ✗

ODEs



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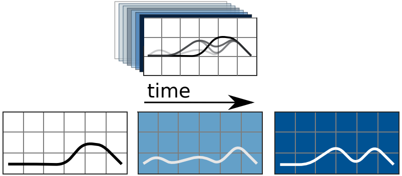

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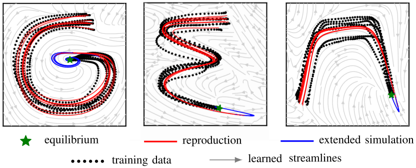

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Learning in the best of both worlds?

responses $\in$ span(*time-invariant* **behaviors**) $\in$ **RKHS**

Linear & Dynamics-invariant RKHS for Learning Dynamics

Koopmanism

Symbol $\mathcal{K}$: the **time-shift** or **Koopman** operator, so $\mathcal{K}y(k) = y(k+1)$.

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$$\mathcal{K}^k \phi_j = \lambda_j^k \phi_j$$

$\rightarrow$ evolve in time with *amplitude* $|\lambda_j|$ & *frequency* $\arg(\lambda_j)$

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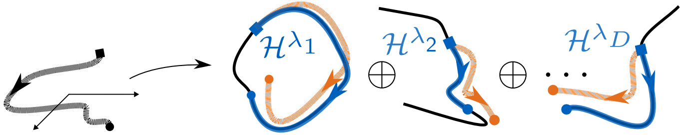
Ours

invariant RKHSs $\{\mathcal{H}^{\lambda_j}\}$

Existing

[Kostic+ 2022; Klus+ 2020]



Koopman Kernel Regression

Koopman kernel (ridge) regression

Given N i.i.d. state-output trajectories $\{x^{(i)}, y^{(i)}\}_{i=1}^N \in (\mathcal{X}, \mathcal{Y})^N$ of length H, compute

Koopman kernel

$$\hat{M} = \underset{M \in \mathcal{S}(\mathcal{H}^{\lambda_1} \oplus \cdots \oplus \mathcal{H}^{\lambda_D})}{\arg\min} \left(\frac{1}{N} \sum_{i \in [N]} \|y^{(i)} - M(x^{(i)})\|_{\mathcal{Y}}^2 + \gamma \|M\|_{\mathcal{H}}^2 \right) \in \operatorname{span}\{ K (\cdot, x^{(i)})\}$$

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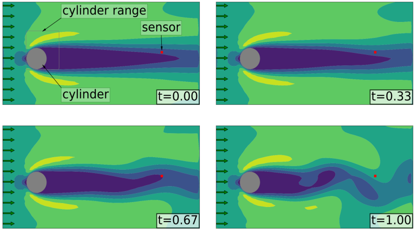
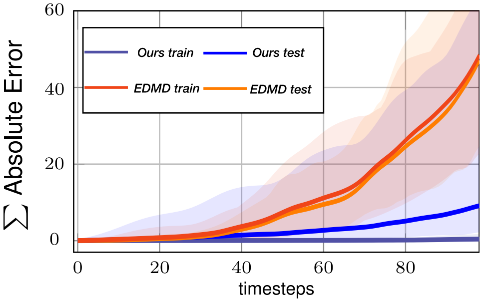
rank-independent generalization

Practical Implications

Flow past a cylinder

Forecasting velocity magnitude of sensor from 50×100-dimensional initial conditions.





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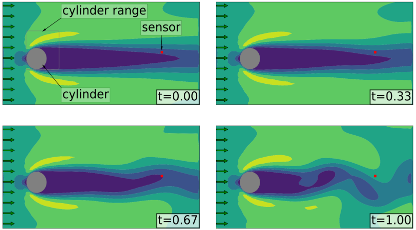
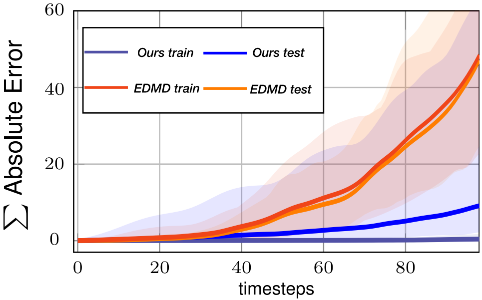


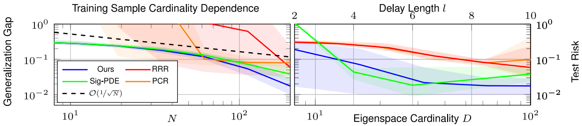


Orders-of-magnitude greater accuracy (also for a wide range of hyperparameters)

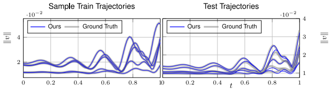
Also in the Full Paper...

Including but not limited to

- full theoretical results
- numerical validation
- extensive comparisons
- complexity analysis
- forecasting and training times



#data $= N \times H = 200 \times 14$	KKR	PCR	RRR	Sig-PDE
Training [s]/Forecast [ms]	**8.0/ 54**	90/ 84	88/150	8.6/5900



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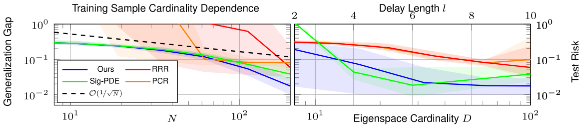

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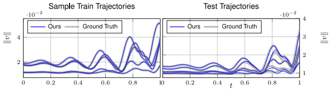

Spoiler: superior to *Koopman operator regression* across the board

References

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