Learning Large-Scale MTP_2 Gaussian Graphical Models via Bridge-Block Decomposition

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MTP_2 Gaussian Graphical Model

Gaussian Graphical Models (GGMs) are a powerful tool for representing complex multivariate data.

- Each node corresponds to one variable
- The lack of an edge between two nodes signifies that the corresponding variables are conditionally independent given the other variables.

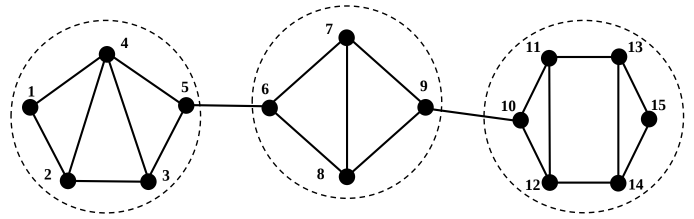
Multivariate Total Positivity of Order 2 (MTP_2) is a property that characterizes a specific type of **positive dependency** among variables.

Learning MTP_2 GGMs from data:

$$\begin{aligned} \underset{\boldsymbol{\Theta}}{\text{minimize}} \quad & -\log\det(\boldsymbol{\Theta}) + \langle \boldsymbol{\Theta}, \boldsymbol{S} \rangle + \textstyle\sum_{i \neq j} \Lambda_{ij} \left| \Theta_{ij} \right|, \\ \text{subject to} \quad & \boldsymbol{\Theta} \succ \mathbf{0} \text{ and } \Theta_{ij} \leq 0, \forall i \neq j. \end{aligned}$$

Bridge-Block Decomposition on Thresholded Graph

- **Thresholded matrix** $T_{ij} = \begin{cases} S_{ij} - \Lambda_{ij} & \text{if } i \neq j \text{ and } S_{ij} > \Lambda_{ij}, \\ 0 & \text{otherwise.} \end{cases}$
- **Thresholded graph**: $(i, j) \in \mathcal{E}$ if $T_{ij} \neq 0$.



- **Bridge**: The edges whose deletion increases the number of graph components.
- **Bridge-Block Decomposition:** Commponents after all bridges are removed.

Main Results

Theorem

Given the bridge-block decomposition of the thresholded graph as $\mathcal{P}^{bbd}$, and the optimal solution of each sub-problem as $\widehat{\boldsymbol{\Theta}}_k$, the optimal solution $\boldsymbol{\Theta}^\star$ can be obtained as

$$\boldsymbol{\Theta}^\star_{i,j} = \begin{cases} [\widehat{\boldsymbol{\Theta}}_k]_{\pi(i),\pi(i)} + \zeta_i & \text{if } i = j \in \mathcal{V}_k, \\ [\widehat{\boldsymbol{\Theta}}_k]_{\pi(i),\pi(j)} & \text{if } i \neq j \text{ and } i, j \in \mathcal{V}_k, \\ -T_{ij} \Big/ (S_{ii} S_{jj} - T_{ij}^2) & \text{if } (i, j) \in \mathcal{B}, \\ 0 & \text{otherwise.} \end{cases}$$

in which $\zeta_i = \frac{1}{S_{ii}} \sum_{(i,m)\in\mathcal{B}} \frac{T_{im}^2}{S_{ii}S_{mm} - T_{im}^2}$ *and* $\zeta_i = 0$ *if* $\forall m : (i, m) \notin \mathcal{B}$.

Proposed Solving Frameworks

1. **Preprocessing:**
   - Compute the thresholded graph.
   - Compute the bridges in thresholded graph.
   - Compute the bridge block decomposition as *clusters*.

2 .**Solving Sub-problems individually:**
   - For each cluster, solve the reduced-size sub-problem.

3. **Obtaining Optimal Solution:**
   - Using proposed theorem to obtain the optimal solution:

$$\mathbf{\Theta}^{\star}_{i,j} = \begin{cases} [\widehat{\mathbf{\Theta}}_k]_{\pi(i),\pi(i)} + \zeta_i & \text{if } i = j \in \mathcal{V}_k, \\ [\widehat{\mathbf{\Theta}}_k]_{\pi(i),\pi(j)} & \text{if } i \neq j \text{ and } i,j \in \mathcal{V}_k, \\ -T_{ij} \Big/ (S_{ii}S_{jj} - T_{ij}^2) & \text{if } (i,j) \in \mathcal{B}, \\ 0 & \text{otherwise.} \end{cases}$$

Data Experiments

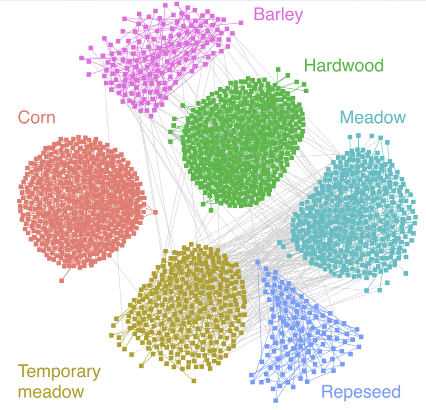


Figure: Local structure.

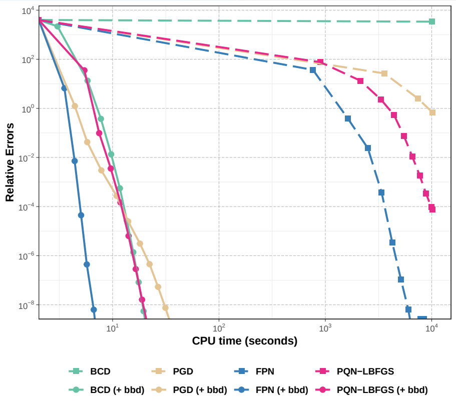


Figure: Convergence results.

Reproducibility

- The code for the experiments can be found at:
  https://github.com/Xiwen1997/mtp2-bbd

- Convex Research Group at HKUST:
  https://www.danielppalomar.com
  https://github.com/dppalomar

THANK YOU!