Distributionally Robust Linear Quadratic Control

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NEURAL INFORMATION PROCESSING SYSTEMS

> *"Life can only be understood going backwards, but it must be lived going forwards."*
> - Søren Kierkegard

Linear Quadratic Control

Linear dynamical system: $x_{t+1} = Ax_t + Bu_t + w_t$

Quadratic cost: $\mathbb{E}_{\mathbb{P}}\left[\sum_{t=0}^{T-1}(x_t{}^\top Q x_t + u_t{}^\top R u_t) + x_T{}^\top Q x_T\right]$

Imperfect state measurements: $y_t = Cx_t + v_t$

Causal controllers: $u_t = \varphi_t(y_0, \ldots, y_t)$

$$\begin{aligned} \min_{u,x,y}\ & \mathbb{E}_{\mathbb{P}}\left[x^\top \mathcal{Q} x + u^\top \mathcal{R} u\right] \\ \text{s.t.}\ & x = \mathcal{H}u + \mathcal{G}w,\ y = \mathcal{C}x + v,\ u \in \mathcal{U}_y \end{aligned}$$

Separation of Estimation and Control

$\mathbb{P} =$ joint distribution of $(x_0, \{w_t\}_{t=0}^{T-1}, \{v_t\}_{t=1}^{T-1})$ ➡ independent with zero mean

Assumption: $\mathbb{P}$ is an arbitrary distribution

Optimal inputs: $u_t^\star = K_t\, \mathbb{E}_{\mathbb{P}}[x_t | y_0, \ldots, y_t]$

feedback gain matrix:
- independent of $\mathbb{P}$
- efficiently computable via DP[1]

state estimator:
- nonlinear in $y_0, \ldots, y_t$
- #P-hard[2]

Assumption: $\mathbb{P}$ is **Gaussian**

state estimator:
- **linear** in $y_0, \ldots, y_t$
- efficiently computable via Kalman filtering

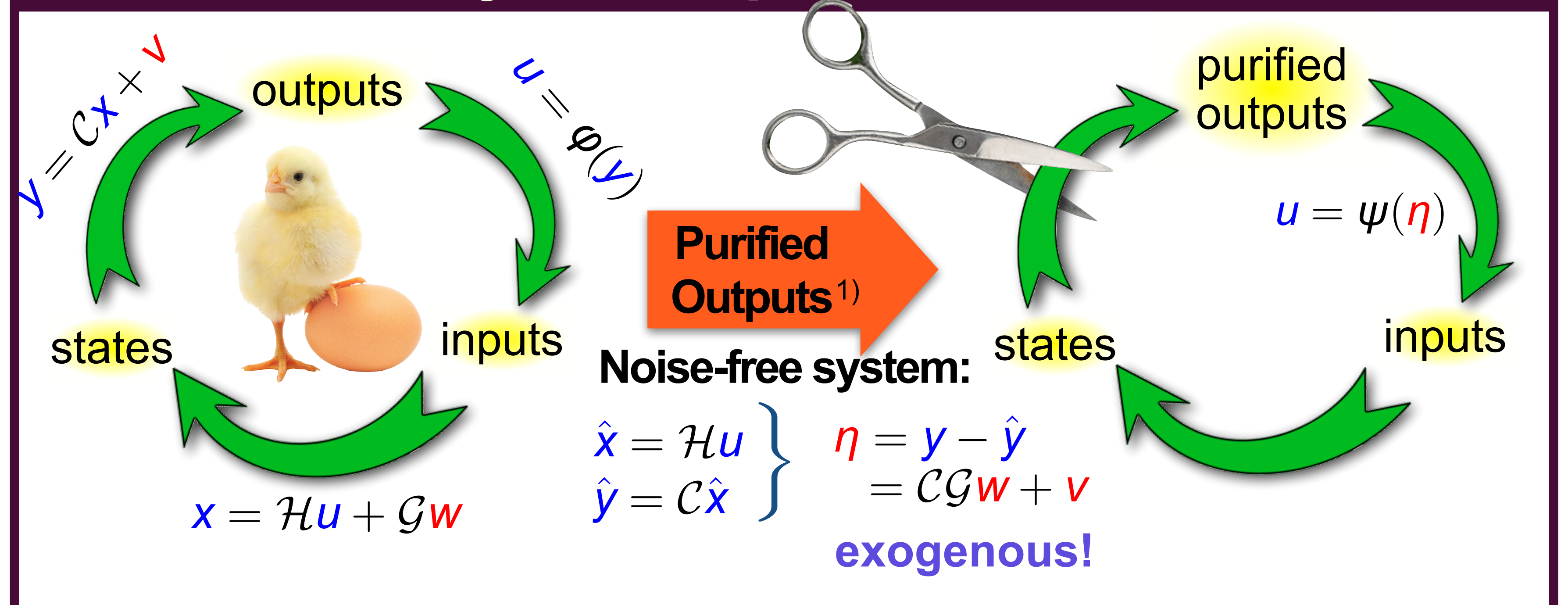
Optimal Transport Problem

$$\mathbb{W}_p(\mathbb{P}_1, \mathbb{P}_2) = \left(\min_{\pi \in \Pi(\mathbb{P}_1, \mathbb{P}_2)} \int \|\xi_1 - \xi_2\|^p \, \mathrm{d}\pi(\xi_1, \xi_2)\right)^{\frac{1}{p}}$$

✓ **Gaussian:** $\mathbb{P}_1 \sim \mathcal{N}(0, \Sigma_1)$, $\mathbb{P}_2 \sim \mathcal{N}(0, \Sigma_2)$ ➡ $\mathbb{W}_2(\mathbb{P}_1, \mathbb{P}_2) = \sqrt{\operatorname{tr}\left[\Sigma_1 + \Sigma_2 - 2(\Sigma_1^{\frac{1}{2}} \Sigma_2 \Sigma_1^{\frac{1}{2}})^{\frac{1}{2}}\right]}.$

Non-Gaussain: ➡ $\mathbb{W}_2(\mathbb{P}_1, \mathbb{P}_2) \geq \sqrt{\operatorname{tr}\left[\Sigma_1 + \Sigma_2 - 2(\Sigma_1^{\frac{1}{2}} \Sigma_2 \Sigma_1^{\frac{1}{2}})^{\frac{1}{2}}\right]}.$

Cyclic Dependencies



Noise-free system:

$$\left.\begin{aligned} \hat{x} &= \mathcal{H}u \\ \hat{y} &= \mathcal{C}\hat{x} \end{aligned}\right\} \quad \begin{aligned} \eta &= y - \hat{y} \\ &= \mathcal{C}\mathcal{G}w + v \end{aligned}$$

exogenous!

causal output-feedback policies $\overset{2)}{=}$ causal **purified** output-feedback policies

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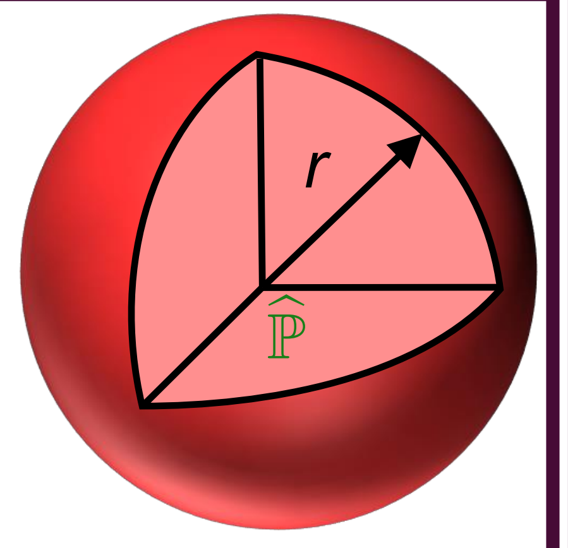
[1] Ben-Tal, Boyd & Nemirovski, *Math. Program.*, 2005.

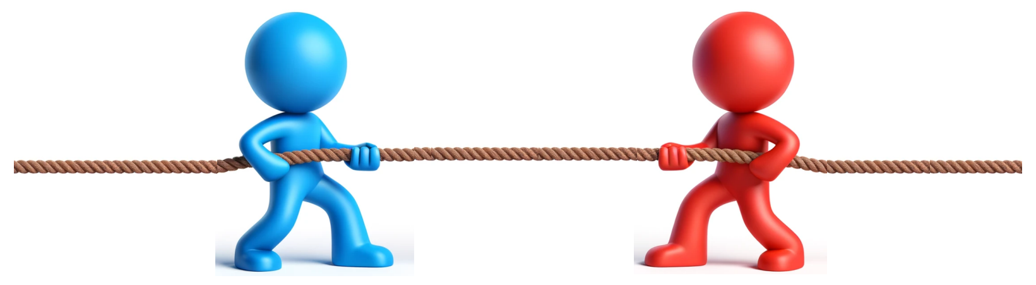
[2] Hadjiyiannis, Goulart & Kuhn, *IEEE Trans. Automat. Contr.*, 2011.

Distributionally Robust LQG Control

$$p^\star = \begin{cases} \min\limits_{u,x}\ \max\limits_{\mathbb{P} \in \mathcal{W}}\ \mathbb{E}_{\mathbb{P}}\left[x^\top \mathcal{Q} x + u^\top \mathcal{R} u\right] \\ \text{s.t.}\quad x = \mathcal{H}u + \mathcal{G}w,\ u \in \mathcal{U}_\eta \end{cases}$$

(u, x)-player moves first





Ambiguity set: $\mathcal{W} = \mathcal{W}_{x_0} \times (\times_{t=0}^{T-1} \mathcal{W}_{w_t}) \times (\times_{t=0}^{T-1} \mathcal{W}_{v_t})$

Gaussian with mean 0

$$\mathcal{W}_{x_0} = \left\{\mathbb{P}_{x_0} \text{ distribution of } x_0 \text{ with mean } 0 \,\middle|\, \mathbb{W}_2(\mathbb{P}_{x_0}, \hat{\mathbb{P}}_{x_0}) \leq r_{x_0}\right\}$$

$$\mathcal{W}_{w_t} = \left\{\mathbb{P}_{w_t} \text{ distribution of } w_t \text{ with mean } 0 \,\middle|\, \mathbb{W}_2(\mathbb{P}_{w_t}, \hat{\mathbb{P}}_{w_t}) \leq r_{w_t}\right\}$$

$$\mathcal{W}_{v_t} = \left\{\mathbb{P}_{v_t} \text{ distribution of } v_t \text{ with mean } 0 \,\middle|\, \mathbb{W}_2(\mathbb{P}_{v_t}, \hat{\mathbb{P}}_{v_t}) \leq r_{v_t}\right\}$$

Structural Results

(a) The primal problem is solved by a **linear policy**;

(b) The dual problem is solved by a **Gaussian distribution**;

(c) **Strong duality** holds.

Proof Sketch:

$$\underline{d}^\star \leq d^\star \leq p^\star \leq \overline{p}^\star$$

Step 3: $\underline{d}^\star = \overline{p}^\star$; Step 2: $\underline{d}^\star \leq d^\star$; Step 1: $p^\star \leq \overline{p}^\star$

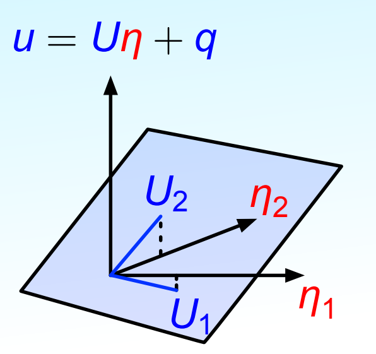
$$d^\star = \begin{cases} \max\limits_{\mathbb{P} \in \mathcal{W}}\ \min\limits_{u,x}\ \mathbb{E}_{\mathbb{P}}\left[x^\top \mathcal{Q} x + u^\top \mathcal{R} u\right] \\ \text{s.t.}\quad x = \mathcal{H}u + \mathcal{G}w,\ u \in \mathcal{U}_\eta \end{cases}$$

$\mathbb{P}$-player moves first

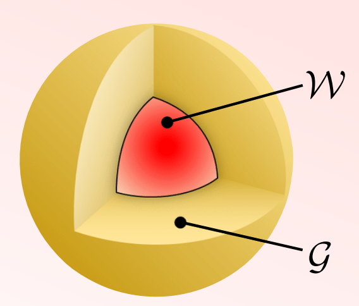
Step 1:

$$\overline{p}^\star = \begin{cases} \min\limits_{u,x,U,q}\ \max\limits_{\mathbb{P} \in \mathcal{G}}\ \mathbb{E}_{\mathbb{P}}\left[x^\top \mathcal{Q} x + u^\top \mathcal{R} u\right] \\ \text{s.t.}\quad x = \mathcal{H}u + \mathcal{G}w,\ u = U\eta + q,\ U \in \mathcal{U} \end{cases}$$

Restriction to linear policies



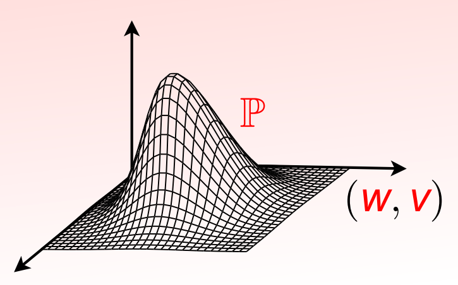
Relaxation to Gelbrich ambiguity set



Step 2:

$$\underline{d}^\star = \begin{cases} \max\limits_{\mathbb{P} \in \mathcal{W}_\mathcal{N}}\ \min\limits_{u,x,U,q}\ \mathbb{E}_{\mathbb{P}}\left[x^\top \mathcal{Q} x + u^\top \mathcal{R} u\right] \\ \text{s.t.}\quad x = \mathcal{H}u + \mathcal{G}w,\ u = U\eta + q,\ U \in \mathcal{U} \end{cases}$$

Restriction to normal distributions



Step 3: Sion's minimax theorem

$$\overline{p}^\star = \begin{cases} \min\limits_{\substack{q \in \mathbb{R}^{pT} \\ U \in \mathcal{U}}}\ \max\limits_{\substack{W \in \mathcal{G}_W \\ V \in \mathcal{G}_V}}\ g(U, q, W, V) \end{cases}$$

$$\underline{d}^\star = \begin{cases} \max\limits_{\substack{W \in \mathcal{G}_W \\ V \in \mathcal{G}_V}}\ \min\limits_{\substack{q \in \mathbb{R}^{pT} \\ U \in \mathcal{U}}}\ g(U, q, W, V) \end{cases}$$

➡ $\underline{d}^\star = \overline{p}^\star$ ➡ $\underline{d}^\star = d^\star = p^\star = \overline{p}^\star$

Numerical Solution to DR-LQG

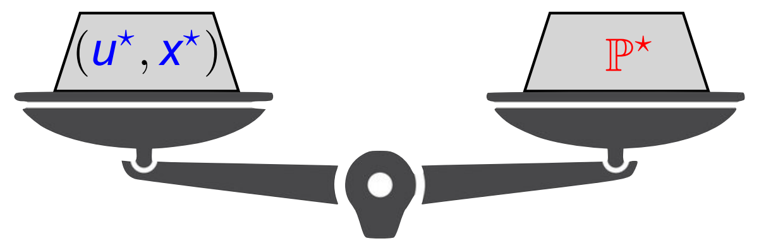
$$\min_{\substack{q \in \mathbb{R}^{pT} \\ U \in \mathcal{U}}}\ \boxed{\max_{\substack{W \in \mathcal{G}_W \\ V \in \mathcal{G}_V}}\ \begin{aligned} &\operatorname{Tr}\left((\mathcal{D}^\top U^\top (\mathcal{R} + \mathcal{H}^\top \mathcal{Q} \mathcal{H}) U \mathcal{D} + 2\mathcal{G}^\top \mathcal{Q} \mathcal{H} U \mathcal{D} + \mathcal{G}^\top \mathcal{Q} \mathcal{G}) W\right) \\ &+ \operatorname{Tr}\left((U^\top (\mathcal{R} + \mathcal{H}^\top \mathcal{Q} \mathcal{H}) U) V\right) + q^\top (\mathcal{R} + \mathcal{H}^\top \mathcal{Q} \mathcal{H}) q \end{aligned}}$$

Convert "max" to "min" via duality

★ SDP with $\mathcal{O}(T(mp + n^2 + p^2))$ variables

⊕ **Theory** ⊖ **Practice**

Nash Equilibrium of Zero-Sum Game



Nash equilibrium

$(u^\star, x^\star)$ solves classic LQG problem under $\mathbb{P}^\star$

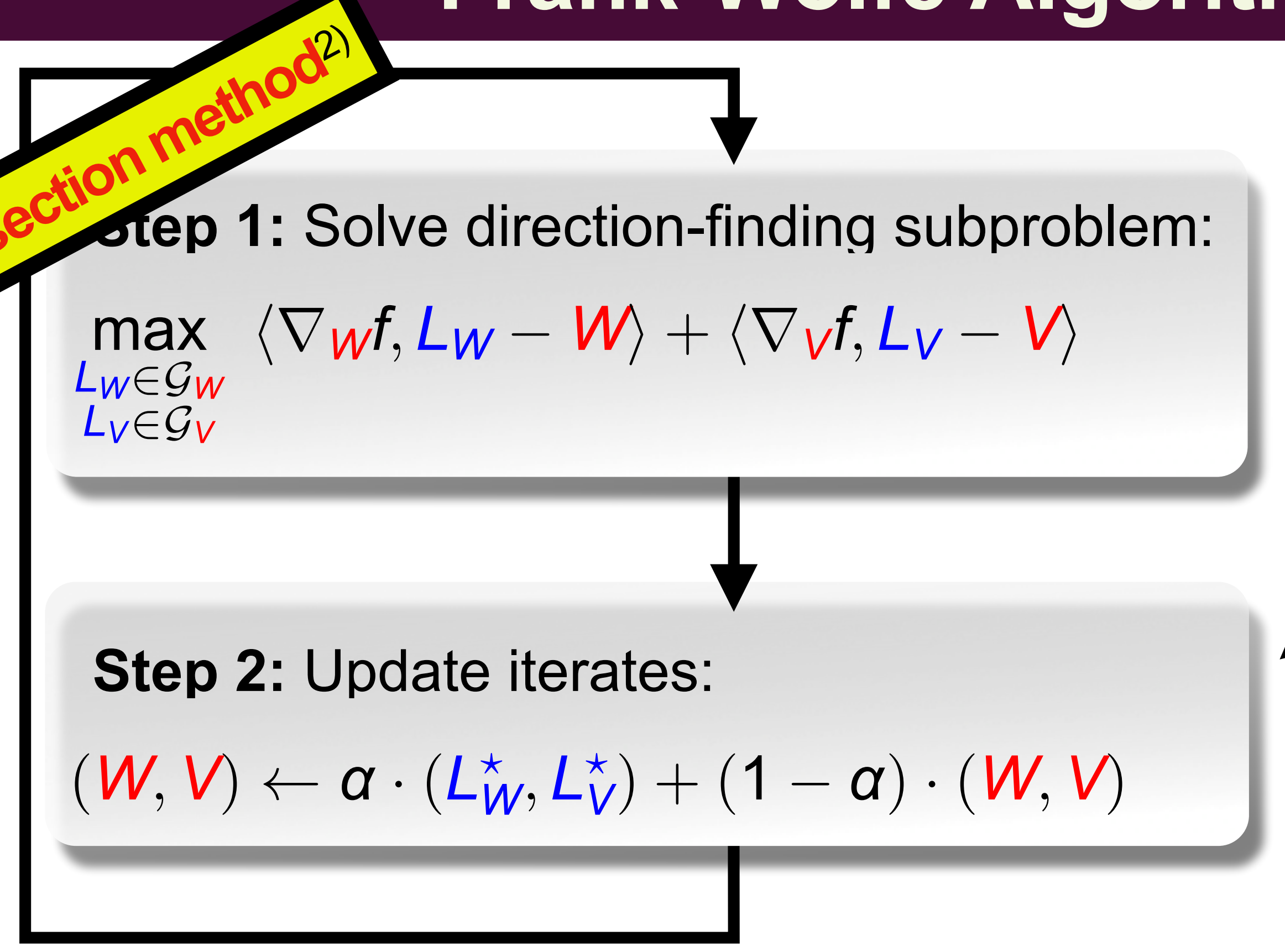
$$u_t^\star = K_t\, \mathbb{E}_{\mathbb{P}^\star}[x_t | y_0, \ldots, y_t] \quad \forall t \in [T-1]$$

computed offline · computed online

$f(W, V)$

$$\max_{\substack{W \in \mathcal{G}_W \\ V \in \mathcal{G}_V}}\ \boxed{\min_{\substack{q \in \mathbb{R}^{pT} \\ U \in \mathcal{U}}}\ \begin{aligned} &\operatorname{Tr}\left((\mathcal{D}^\top U^\top (\mathcal{R} + \mathcal{H}^\top \mathcal{Q} \mathcal{H}) U \mathcal{D} + 2\mathcal{G}^\top \mathcal{Q} \mathcal{H} U \mathcal{D} + \mathcal{G}^\top \mathcal{Q} \mathcal{G}) W\right) \\ &+ \operatorname{Tr}\left((U^\top (\mathcal{R} + \mathcal{H}^\top \mathcal{Q} \mathcal{H}) U) V\right) + q^\top (\mathcal{R} + \mathcal{H}^\top \mathcal{Q} \mathcal{H}) q \end{aligned}}$$

Frank-Wolfe Algorithm [1]

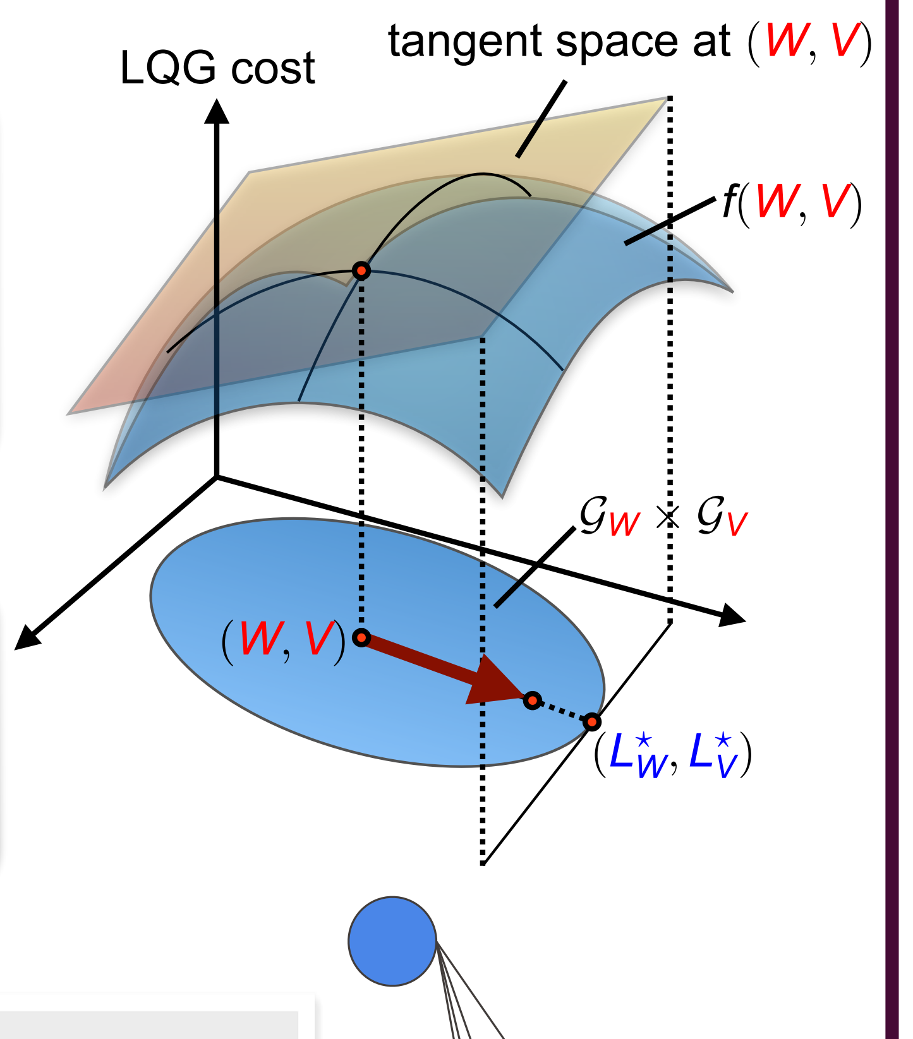


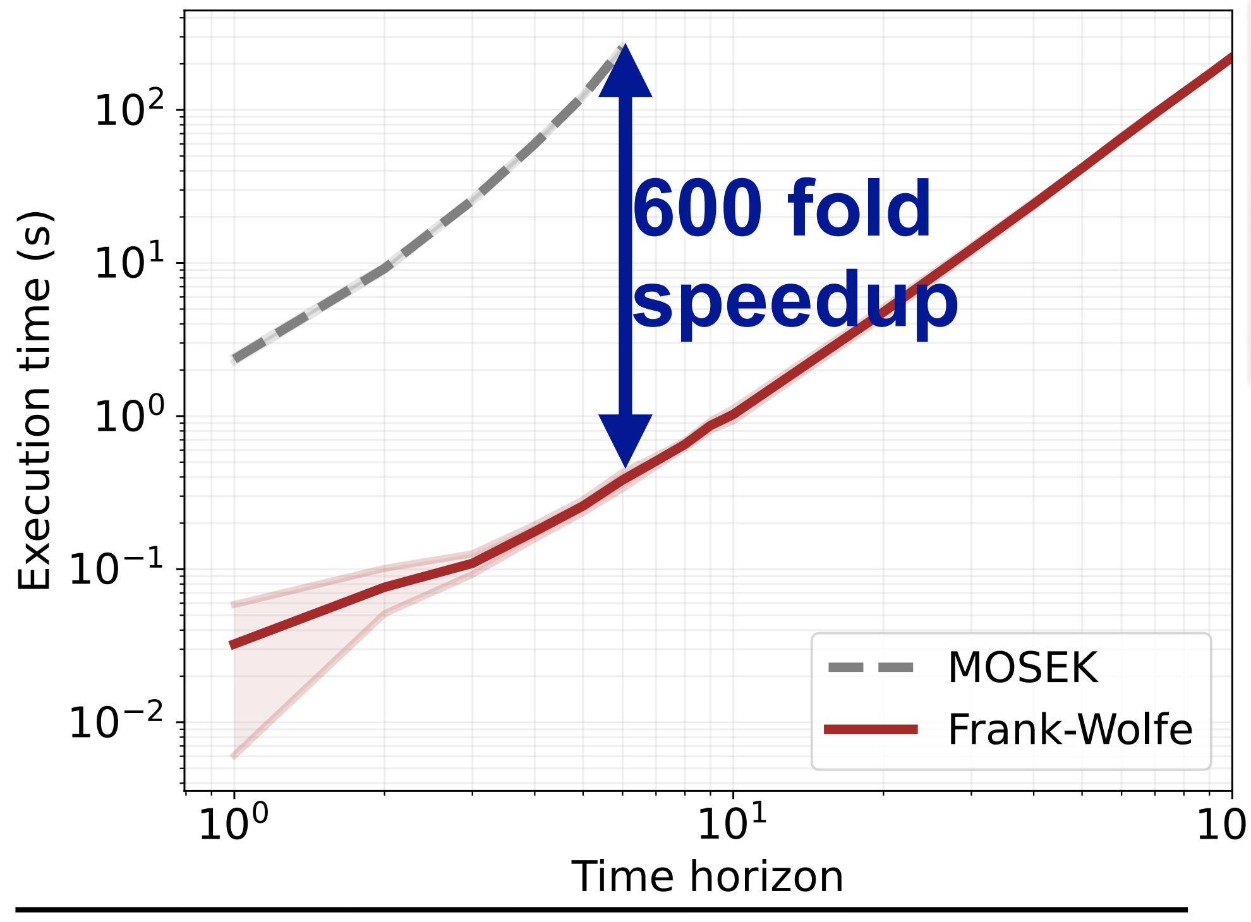
Step 1: Solve direction-finding subproblem:

$$\max_{\substack{L_W \in \mathcal{G}_W \\ L_V \in \mathcal{G}_V}}\ \langle \nabla_W f, L_W - W \rangle + \langle \nabla_V f, L_V - V \rangle$$

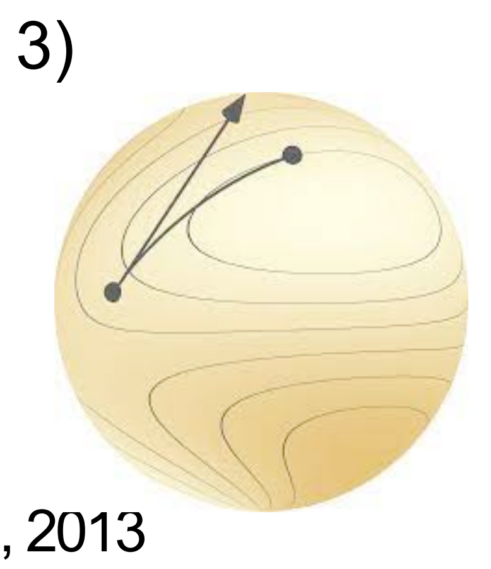
Step 2: Update iterates:

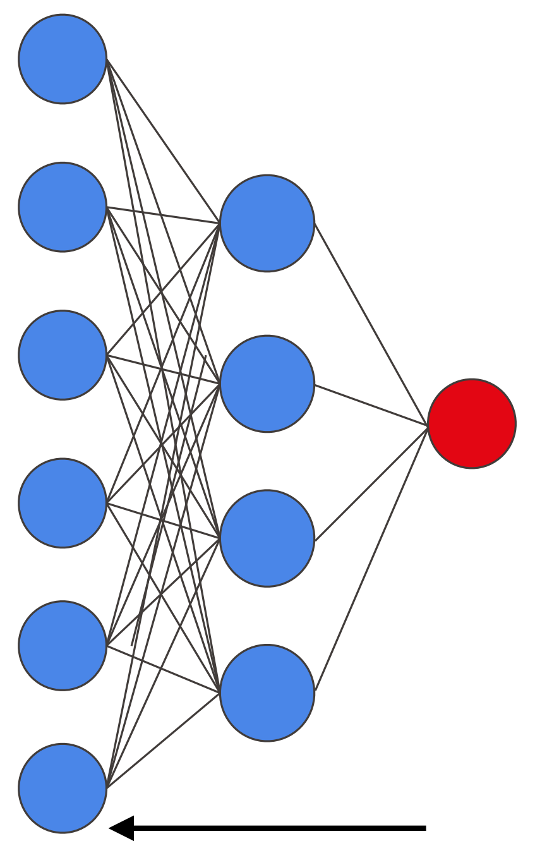
$$(W, V) \leftarrow \alpha \cdot (L_W^\star, L_V^\star) + (1 - \alpha) \cdot (W, V)$$





10 inputs
10 outputs
10 states





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[1] Frank & Wolfe, *Nav. Res. Logist.*, 1956; Jaggi, *ICML*, 2013

[2] Nguyen, Shafieezadeh-Abadeh, Kuhn & Mohajerin Esfahani, *Math. Oper. Res.*, 2023

[3] Townsend, Koep & Weichwald, *JMLR*, 2016